UNIT – 3

LATTICE

∆*X* = {(*x, x* )|*x* ∈ *X* }

Definition 1

On a set X, ≤ ⊆ X × X is a partial order if

e *reflexive:* ∆*X* ⊆ ≤

e *anti-symmetric:* ≤ ∩ ≤*−*1⊆ ∆*X*

e *transitive:* ≤ ◦ ≤ ⊆ ≤

We will use *x* ≤ *y* to denote (*x, y* ) ∈ ≤. Let *x < y* ¾(*x* ≤ *y* ∧ *x* ƒ= *y* ).

### Example 1

Is the following a partial order on {a, b, c}?

e ≤= {(*a, a*)*,* (*b, b*)*,* (*c, c*)}

#### e ≤= ∅

e ≤= {(*a, a*)*,* (*b, b*)*,* (*c, c*)*,* (*a, b*)}

e ≤= {(*a, a*)*,* (*b, b*)*,* (*c, c*)*,* (*a, b*)*,* (*b, a*)}

 e ≤= {(*a, a*)*,* (*b, b*)*,* (*c, c*)*,* (*a, b*)*,* (*b, c*)}

### Definition 2

A poset (X, ≤) is a set equipped with partial order ≤ on X

### Example 2

Is the following a partial order?

e (N*,* ≤)

e (N × N*,* {( (*a, b*)*,* (*c, d* ) )|*a* ≤ *c*})

e (N × N*,* {( (*a, b*)*,* (*c, d* ) )|*a* ≤ *c* ∧ *b* ≤ *d* })

e ({*a, b, c*}*,* {(*a, a*)*,* (*b, b*)*,* (*c, c*)*,* (*a, b*)*,* (*a, c*)})

### Definition 3

The covering relation « for poset (X, ≤) is

*x* « *y* ¾ (*x < y* ) ∧ ¬(∃*z.x < z* ∧ *z < y* )

#### In other words, « contains only immediate parents and has no self-edges.

Example 3

Consider poset ({a, b, c, d, e}, ≤), where

≤ = {(*a, a*)*,* (*b, b*)*,* (*c, c*)*,* (*d, d* )*,* (*e, e*)*,* (*a, b*)*,* (*a, c*)*,* (*a, d* )*,* (*b, e*)*,* (*c, e*)*,* (*d, e*)*,* (*a, e*)}

Therefore,

« = {(*a, b*)*,* (*a, c*)*,* (*a, d* )*,* (*b, e*)*,* (*c, e*)*,* (*d, e*)}

#### We draw posets (*X,* ≤) as DAG. Nodes are from *X* and edges are from «.

DAG will be vertically aligned, i.e., if there is an edge between *x* and *y* , and

*x* is located below *y* then *x* « *y* .

### Example 4

Let us consider again our previous poset ({a, b, c, d, e}, ≤), where

« = {(*a, b*)*,* (*a, c*)*,* (*a, d* )*,* (*b, e*)*,* (*c, e*)*,* (*d, e*)}

#### Nodes at same level are incomparable.

Definition 4

*For a poset* (*X,* ≤)*, C* ⊆ *X is chain if* ∀*x, y* ∈ *C. x* ≤ *y* ∨ *y* ≤ *x*

### Definition 5

*For a poset* (*X,* ≤)*, C* ⊆ *X is antichain if* ∀*x, y* ∈ *C. x* ≤ *y* ⇒ *y* = *x*

### Example 5

antichain

*chain*

### Definition 6

A poset (X, ≤) satisfies ascending chain condition if for any sequence x0 ≤ x1 ≤ x2 ≤ . . . , ∃k.∀n > k xk = xn

#### Symmetrically, we define descending chain condition

Definition 7

A poset (X, ≤) is called well ordered if it satisfies descending chain condition

### Example 6

(N, ≤) satisfies descending chain condition but not ascending chain condition.

### Definition 8

*For poset* (*X,* ≤) *and S* ⊆ *X, minimal* (*S*) ¾{*x* ∈ *S*|¬∃*y* ∈ *S. y < x* }

*maximal* (*S*) ¾{*x* ∈ *S*|¬∃*y* ∈ *S. y > x* }

### Definition 9

*For poset* (*X,* ≤) *and S* ⊆ *X,*

*min*(*S*) ¾ *x if* {*x* } = *minimal* (*S*) *//min*(*S*) *may not exist*

*max* (*S*) ¾ *x if* {*x* } = *maximal* (*S*)

#### Consider poset (*X,* ≤)

If *min*(*X* ) exists, we denote *min*(*X* ) by ⊥

If *max* (*X* ) exists, we denote *max* (*X* ) by T

### Definition 10

*For poset* (*X,* ≤) *and S* ⊆ *X,*

e *x* ∈ *X is upper bound of S if* ∀*y* ∈ *S. y* ≤ *x*

e *x* ∈ *X is lower bound of S if* ∀*y* ∈ *S. x* ≤ *y*

### Definition 11

x ∈ X is least upper bound(lub) of S if x is upper bound of S and

∀*u.* (∀*y* ∈ *S. y* ≤ *u*) ⇒ *x* ≤ *u lub is usually denoted by* ∨*,* H *(called join).*

### Definition 12

x ∈ X is greatest lower bound(glb) of S if x is lower bound of S and

∀*u.* (∀*y* ∈ *S. u* ≤ *y* ) ⇒ *u* ≤ *x lub is usually denoted by* ∧*,* H *(called meet).*

Note: lub and glb may not exist.

### Theorem 1

Note that the uniqueness is not obvious by the definition of H

For poset (X, ≤) and S ⊆ X, if HS exists then it is unique.

### Proof.

#### e Suppose *x* and *y* are H*S*.

e By definition of H, *x* and *y* both are upper bounds of *S*.

e Since *x* is upper bound and *y* is H*S*, therefore *y* ≤ *x* .

e Symmetrically, *x* ≤ *y* .

e Due to anti-symmetry, *x* = *y* .

Therefore, H and H are partial functions : 2*X ‹*→ *X*

e If *S* = {*x, y* }, we will write *x* H *y*

#### e The infix usage usually means, lub of finite elements

Definition 13

A join semi-lattice (X, ±, H) is a poset (X, ±) such that

∀*x, y* ∈ *X. x* H *y exists.*

### Definition 14

A meet semi-lattice (X, ±, H) is a poset (X, ±) such that

∀*x, y* ∈ *X. x* H *y exists.*

### Example 8

N2 is a meet semi-lattice.

### Theorem 2

A join semi-lattice (X, ±, H) satisfies

e (*a* H *b*) H *c* = *a* H (*b* H *c*) *(associativity)*

e (*a* H *b*) = (*b* H *a*) *(commutativity)*

e *a* = (*a* H *a*) *(idempotence)*

**Commentary:** Please read the theorem carefully. It says that the three conditions characterizes semi-lattices.

We also need to show H is lub with respect to ±.

### Theorem 3

Let X be a set with function H : X × X → X satisfying

(*a* H *b*) H *c* = *a* H (*b* H *c*)*,* (*a* H *b*) = (*b* H *a*)*, and* (*a* H *a*) = *a.*

*Let a* ± *b* ¾ (*a* H *b*) = *b.*

Then, (X, ±, H) is a join semi-lattice.

### Proof.

#### We need to show that ± is a partial order, i.e.,

e ± is reflexive,

e ± is transitive, and

e ± is anti-symmetric.

Proof(contd.)

**claim:** ± is reflexive

e *a* ± *a* holds because (*a* H *a*) = *a*.

**claim:** ± is transitive

1. Assume *a* ± *b* and *b* ± *c*

#### def of ±,

(*a* H *b*) = *b* and (*b* H *c*) = *c*.

#### By substitution, ((*a* H *b*) H *c*) = *c*.

1. Due to associativity, *a* H (*b* H *c*) = *c*.
2. Due to 4, *a* H *c* = *c*.(why?)

#### Therefore *a* ± *c*. ...

Proof(contd.)

**claim:** ± is anti-symmetric

1. Assume *a* ± *b* and *b* ± *a*
2. By def of ±, (*a* H *b*) = *b* and (*b* H *a*) = *a*

#### By commutativity, *a* = *b*

**claim:** H is lub

1. *b* ± *a* H *b*, because *a* H (*a* H *b*) = (*a* H *a*) H *b* = *a* H *b*.
2. Similarly, *a* ± *a* H *b*.
3. Let *x* be such that *a* ± *x* and *b* ± *x* .
4. Therefore, (*a* H *x* ) = *x* = (*b* H *x* )

#### After substitution, (*a* H (*b* H *x* )) = *x*

1. Apply associativity, ((*a* H *b*) H *x* ) = *x* , which is (*a* H *b*) ± *x*

 7. Therefore, *a* H *b* = *lub*({*a, b*})

#### We write (*X,* ±*,* H) to describe a semi-lattice.

Due to the previous theorem, if we know something is a semi-lattice, we need not write both the second and third component.

One defines the other.

We may only write (*X,* H).

Definition 15

exist for finite sets

H and H are forced to

A lattice (X, ±, H, H) is a poset (X, ±) such that

∀x, y ∈ X both x H y and x H y exist.

#### Properties of lattice

1. (*a* H *b*) H *c* = *a* H (*b* H *c*) (associativity)
2. (*a* H *b*) H *c* = *a* H (*b* H *c*)

#### (*a* H *b*) = (*b* H *a*) (commutativity)

1. (*a* H *b*) = (*b* H *a*)

#### (*a* H *a*) = *a* (idempotence)

1. (*a* H *a*) = *a*)
2. *a* H (*a* H *b*) = *a* (absorption)
3. *b* H (*a* H *b*) = *b*

#### Properties 1-6 were already present in semi-lattices. The above properties are axiomatization of lattice Observe that distributivity is missing!!!

Definition 16

A complete partial order(cpo) is a poset (X, ±) such that every increasing chain in X has a lub in X

### Definition 17

A complete lattice is a poset (X, ±) such that for all S ⊆ X has HS in X.

### Example 9

N2 is not a complete lattice.

### Theorem 4

Let (X, ±) be a complete lattice.

1. *complete lattice has* ⊥
2. *complete lattice has* T

### Proof.

1. ⊥ = H∅(why?)
2. ⊥ = H*X* (why?)

### Theorem 5

Let (X, ±) be a complete lattice. For all S ⊆ X, HS exists.

### Proof.

**claim:** H*S* = H{*y* |∀*x* ∈ *S. y* ± *x* }

*S*

∈ *ub*(*Y* )

*lb*(*S*) sH*Y*

*Y* = {*y* |∀*x* ∈ *S. y* ± *x* }

∈ *lb*(*S*)

### Definition 18

For a poset (X, ±) with T element, a moore family M ⊆ X is such that

#### e T ∈ *M*

e ∀*S* ⊆ *M.* H *S exists and* H*S* ∈ *M*

### Theorem 6

Let (X, ≤) be a poset with T element. If M ⊆ X is a moore family then

(M, ±, T, HM) is a complete lattice.

### Proof.

#### (*X,* ≤) is poset then (*M,* ≤) is a poset

1. Since ∀*S* ⊆ *M.* H *S* exists, *M* is a complete lattice due to Theorem [4.](#_bookmark2)

We have seen the following objects

cpo

Moore family

Set *X*

Poset (*X,* ±)

Semi-lattice (*X,* ±*,* H) and (*X,* ±*,* H) Lattice (*X,* ±*,* H*,* H)

Complete lattice (*X,* ±*,* T*,* ⊥*,* H*,* H)

# End of Lecture