**Series**

**Series** : Let (*an*) be a sequence of real numbers. Then an expression of the form *a*1+*a*2+*a*3+*::::::*denoted by *;* is called a series.

Examples : 1. 1/ 1 + 1/2 + 1/3 + .... or

2. 1 + 1/4 + 1/9 + .... or

**Partial sums** : *Sn* = *a*1 + *a*2 + *a*3 + *::::::* + *an* is called the nth partial sum of the series

*.*

**Convergence or Divergence of *an:*** If *Sn ! S* for some *S* then we say that the series*an* converges to *S*.

If (*Sn*) does notconverge then we say that the series*an* diverges.

Examples :

1.

diverges because *Sn* = *log*(*n* + 1)*.*

2.

converges because *Sn* = 1-1/*n*+1 1*.*

3. If 0 *< x <* 1*;* then the geometric series converges to 1/(1*-x)* because

*Sn* = (1*-xn*+1)/(1-*x).*

**Necessary condition for convergence**

**Theorem 1** : *If converges then an* 0*.*

Proof : *Sn*+1- *Sn* = *an*+1 *S - S* = 0*.*

The condition given in the above result is necessary but not sufficient i.e., it is possible that *an* 0 an diverges.

Examples :

1. If 1, then diverges because *an*

2.

diverges because *an0*

3.

diverges, however, *log*((*n*+1)/*n* ) *→* 0*.*

**Necessary and sufficient condition for convergence**

**Theorem 2**: *Suppose an* 0; *n .Then an converges if and only if* (*Sn*) *is bounded above.*

Proof : Note that under the hypothesis, (*Sn*) is an increasing sequence.

Example : The Harmonic series diverges because

*=* 1 + ½ + 2.(1/4) + 4 (1/for all) + *:::* + *(1/)* = 1 + ( *k /* 2) for all k.

**Theorem 3**: *If converges then converges.*

Proof : Since converges the sequence of partial sums of

satisfies the Cauchy criterion. Therefore, the sequence of partial sums of

satisfies the Cauchy criterion.

**Remark** : Note that converges if and only if

converges for any *p* 1.

**Tests for Convergence**

Let us determine the convergence or the divergence of a series by comparing it to one whose behavior is already known.

**Theorem 4 : (Comparison test )** *Suppose* 0 *an bn for n ¸ k for some k: Then*

*(1) The convergence of implies the convergence of*

*(2) The divergence of implies the divergence of .*

Proof : (1) Note that the sequence of partial sums of

is bounded. Apply Theorem 2.

(2) This statement is the contrapositive of (1).

Examples:

1. converges because 1/(*n*+1)(*n*+1) 1/*n*(*n*+1) *:* This implies that converges.

2. diverges because 1/*n*

3. converges because *n*2 *< n*! for *n* > 4*.*

Problem 1 : *Let an* 0*: Then show that both the series*

*and*  *converge or diverge together.*

*Solution :* Suppose

converges. Since 0 by comparison test converges.

Suppose converges. By the Theorem 1, (*an/*1+*an)*0. Hence *an* 0 and therefore

1 1 + *an <* 2 eventually.

Hence 0  *an* 1+*an*. Apply the comparison test.

**Theorem 5 : (Limit Comparison Test)** *Suppose an; bn* 0 *eventually. Suppose L.*

1. *If L* R*;L >* 0*, then both and converge or diverge together.*

2. *If L* R*;L* = 0*, and converges then converges.*

3. *If L* = *1 and diverges then diverges.*

Proof: 1. Since *L >* 0, choose  *>* 0, such that *L - >* 0. There exists such that

0  *L< L+*Use the comparison test.

2. For each  *>* 0, there exists such that 0 *< an/bn*

*<*  *; n > n*0. Use the comparison test.

3. Given *α>* 0, there exists *n*0 such that *an/bn > α n > n*0. Use the comparison test.

Examples :

1. (1 *– n.sin (*1/*n*) converges. Take *bn* = 1/*n*2 in the previous result.

2.

*log*(1 + 1/*n*) converges. Take *bn* = 1/*n*2 in the previous result.

**Theorem 6 (Cauchy Test or Cauchy condensation test**) *If an* 0 *and an*+1  *an n, then*  *converges if and only if*  *converges.*

Proof : Let *Sn* = *a*1 + *a*2 + *::::* + *an* and *Tk* = *a*1 + 2*a*2 + *::::* +*:*

Suppose (*Tk*) converges. For a fixed *n;* choose *k* such that 2*k n*. Then

*Sn* = *a*1 + *a*2 + *::::* + *an*

*· a*1 + (*a*2 + *a*3) + *:::::* + (*a*2*k* + *::::* + *a*2*k*+1*¡*1)

*· a*1 + 2*a*2 + *::::* + 2*ka*2*k*

= *Tk:*

This shows that (*Sn*) is bounded above; hence (*Sn*) converges.

Suppose (*Sn*) converges. For a fixed *k;* choose *n* such that *n* 2*k:* Then

*Sn* = *a*1 + *a*2 + *::::* + *an*

*¸ a*1 + *a*2 + (*a*3 + *a*4)*:::::* + (*a*2*k¡*1+1 + *::::* + *a*2*k* )

*¸* 1

2*a*1 + *a*2 + 2*a*4 + *::::* + 2*k¡*1*a*2*k*

= 1

2*Tk:*

This shows that (*Tk*) is bounded above; hence (*Tk*) converges. ¤

Examples:

1. 1/*n p* converges if *p >* 1 and diverges if *p* 1*:*

2.

1/*n*(*logn*)*p* converges if *p >* 1 and diverges if *p* 1*:*

Problem 2 : Let *an* 0*; an*+1  *an for all n* and suppose converges. Show that *nan* 0 as

*n* .

*Solution :* By Cauchy condensation test

converges. Therefore 2*ka*2*k* 0 and hence 2*k*+1*a*2*k !* 0 as *k ! 1*.

Let 2*k n* 2*k*+1. Then *nan na*2*k*2*k*+1*a*2*k* 0. This implies that

*nan* 0 as *n*

**Theorem 7 (Ratio test)**  *Consider the series ; an not equal to* 0 *for all n:*

*1. If q eventually for some* 0 *< q <* 1*; then converges.*

*2. If* 1 ;*eventually then diverges.*

Proof: 1. Note that for some *N; j an*+1 *j · q j an j for all n ¸ N:* Therefore, *j aN*+*p j · qp j aN j*

*for all p >* 0*:* Apply the comparison test.

2. In this case *j an j* 9 0.

Corollary 1: *Suppose an not equal to* 0 *for all n; and j an*+1

*an*

*j ! L for some L:*

*1. If L <* 1 *then*

*j an j converges.*

*2. If L >* 1 *then*

*an diverges.*

*3. If L* = 1 *we cannot make any conclusion.*

Proof :

1. Note that *j an*+1

*an*

*j < L* + (1*¡L*)

2 eventually. Apply the previous theorem.

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2. Note that *j an*+1

*an*

*j > L ¡* (*L¡*1)

2 eventually. Apply the previous theorem.

Examples :

1. 1/*n*! converges because *an*+1 */ an* 0*.*

2. *nn / n*! diverges because *an*+1 / *an* = (1 + 1/*n*)*n e >* 1*.*

3. 1*/n* diverges and 1/*n*2 converges, however, in both these cases

*an*+1*/an* 1*.*

**Theorem for all : (Root Test )** *If* 0  *an xn or* 0 *an*1*/n x eventually*

*for some* 0 *< x <* 1 *then converges.*

Proof : Immediate from the comparison test.

Corollary 2: *Suppose j an j*1*=n ! L for some L: Then*

*1. If L <* 1 *then*

*j an j converges.*

*2. If L >* 1 *then*

*an diverges.*

*3. If L* = 1 *we cannot make any conclusion.*

Examples :

1. converges because *a*1*/n* = 1/

*logn* 0*:*

2. *n*2 converges because *an*1*/n*= 1

(1+ 1/*n* )*n* 1/*e <* 1*:*

3. diverges and converges, however, in both these cases *a*1*/n* 1*.*

**Theorem 9 : (Leibniz test )** *If* (*an*) *is decreasing and an* 0*, then*

*converges.*

Proof : Note that (*S*2*n*) is increasing and bounded above by *S*1. Similarly, (*S*2*n*+1) is decreasing

and bounded below by *S*2. Therefore both converge. Since *S*2*n*+1 *¡S*2*n* = *a*2*n*+1 *!* 0*;* both (*S*2*n*+1)

and (*S*2*n*) converge to the same limit and therefore (*Sn*) converges.

Examples :

;

and

converge.

Problem 3: *Let fang be a decreasing sequence, an ¸* 0 *and* lim

*n!1*

*an* = 0*: For each n 2* N*; let*

*bn* = *a*1+*a*2+*:::*+*an*

*n : Show that*

(*¡*1)*nbn converges.*

*Solution :* Note that *bn*+1 *¡ bn* = 1

*n*+1(*a*1 + *a*2 + *:::* + *an*+1) *¡* 1

*n*(*a*1 + *:::* + *an*) = *an*+1

*n*+1 *¡* (*a*1+*:::*+*an*)

*n*(*n*+1) .

Since (*an*) is decreasing, *a*1 + *:::* + *an ¸ nan*. Therefore, *bn*+1 *¡ bn · an*+1*¡an*

*n*+1 *·* 0. Hence (*bn*) is

decreasing.

We now need to show that *bn !* 0. For a given *² >* 0, since *an !* 0, there exists *n*0 such that

*an < ²*

2 for all *n ¸ n*0.

Therefore, *j a*1+*¢¢¢*+*an*

*n j* = *j a*1+*¢¢¢*+*an*0

*n* + *an*0+1+*¢¢¢*+*an*

*n j ·j a*1+*¢¢¢*+*an*0

*n j* +*n¡n*0

*n*

*²*

2 . Choose *N ¸ n*0 large

enough so that *a*1+*¢¢¢*+*an*0

*N < ²*

2 . Then, for all *n ¸ N*, *a*1+*¢¢¢*+*an*

*n < ²*. Hence, *bn !* 0. Use the Leibniz

test for convergence.