**Sequences**

**Definition**  A sequence of real numbers is a real-valued function defined on the set of natural numbers, i.e. a function f : N −→ R.

In other words, a sequence can be written as f(1), f(2), f(3), ... We shall denote by an such a sequence where an = f(n) ∈ R. The number an is called the n-th term of the sequence. **Definition** A sequence {an} is said to converge to a real number L if ∀ > 0, ∃N ∈ N such that if n > N, then |an − L| < .

Example 1.1. Prove that the sequence n n + 1 converges to 1, that is, prove that limx→∞ n n + 1 = 1

Proof. We are required to show that for any given > 0, we can find a natural number N such that if n > N, then | n n + 1 − 1| = | −1 n + 1 | < .

We first note that 1 n + 1 < 1 n for any N ∈ N.

Now that for any given , by the Archimedean Property, there exists N ∈ N such that 1 N < . It follows that if n > N, then 1 n + 1 < 1 n < 1 N < and so we are done.

**Some properties of a convergent sequence**

**Theorem** (**Uniqueness of Limits).** Let {an} be a convergent sequence. Then the limit of the sequence is unique.

**Definition**  A sequence {an} is bounded above if there exists M ∈ R such that an < M for all n ∈ N. Similarly, a sequence {an} is bounded below if there exists m ∈ R such that an > m for all n ∈ N.

A sequence {an} is bounded if it is bounded above and below.

**Theorem** (**Boundedness of Convergent Sequences).** If {an} is a convergent sequence, then it is bounded.

**Theorem** (**Arithmetic Properties of Convergent Sequences).** Let {an} and {bn} be two sequences.

Suppose limn→∞ an = A and limn→∞ bn = B for some A, B ∈ R. Then

(a) limn→∞ (an + bn) = limn→∞ an + limn→∞ bn = A + B

(b) limn→∞ (anbn) = ( limn→∞ an)( limn→∞ bn) = AB

(c) limn→∞ can = c limn→∞ an = cA for any c ∈ R.

(d) limn→∞ an/ bn =( limn→∞ an)/ (limn→∞ bn )= A /B ,

provided that bn ≠0 for all n ∈ N and B ≠0.

**Theorem** **(Sandwich Theorem).** Let {an}, {bn} and {cn} be sequences. Suppose that an ≤ bn ≤ cn for all n ∈ N, and that limn→∞ an = limn→∞ cn = L. Then limn→∞ bn = L.

**Monotone Sequences**

**Definition** Let {an} be a sequence. We say that {an} is increasing if an ≤ an+1 for all n ∈ N and is decreasing if an ≥ an+1 for all n ∈ N.

We say that {an} is a monotone if it is either increasing or decreasing.

**Theorem** (a) If {an} is increasing and bounded above, then {an} converges.

(b) If {an} is decreasing and bounded below, then {an} converges.

**Subsequences and the Bolzano-Weiertrass Theorem**

**Definition** . Let {an} be a sequence of real numbers, and let n1 < n2 < n3 < · · · be a strictly increasing sequence of natural numbers.

Then the sequence an1 , an2 , an3 · · · is called a subsequence of {an} and is denoted by {ank }, where k ∈ N indexes the subsequence.

**Theorem** . If {an} is a sequence converging to L, then every subsequence {ank } also converges to L.

**Theorem** . Let {an} be a sequence. The {an} has a monotone subsequence.

Proof. To prove this theorem, we call the k-th term dominant if it is greater than or equal to all the following terms.

In other words, the term ak is dominant if ak ≥ am for all m ≥ k. There are two cases to consider:

Case 1: There are infinitely many dominant terms in the sequence {an}.

Then we have a infinite subsequence an1 , an2 , an3 , . . . Now ani is dominant for all i ∈ N and ni < ni+1, so we must haveani ≥ ani+1 . Similarly, we have an1 ≥ an2 ≥ an3 ≥ · · ·

Hence we have a decreasing subsequence ank and we are done.

Case 2: There are only finitely many dominant terms in the sequence {an}. (including the case where there is no dominant term).

Suppose an is the last dominant term, and let n1 = N + 1. Now an1 is not dominant and so there must exist n2 > n1 such that an1 ≤ an2 .

Continuing this way, we obtain an increasing subsequence an1 , an2 , an3 , · · · as desired.

**Theorem (the Bolzano-Weiertrass Theorem).** Let {an} be a bounded sequence. Then {an} has a convergent subsequence.

**Cauchy’s Convergence Criterion**

**Definition** . A sequence {an} is called a Cauchy sequence if ∀ > 0, ∃N ∈ N such that if n, m > N, then |an − am| < .

**Theorem** . Let {an} be a convergent sequence. Then {an} is a Cauchy sequence.

**Theorem** . Let {an} be a Cauchy sequence. Then {an} is a convergent sequence.