COMPLEX NUMBER

A complex number can be written in the form a + b*i* where a and b are real numbers  (including 0) and *i* is an imaginary number.

Therefore a complex number contains two 'parts':

* one that is real
* and another part that is imaginary

**note:** Even though complex have an imaginary part, there are actually many real life applications of these "imaginary" numbers including [oscillating springs](https://www.mathwarehouse.com/algebra/complex-number/real-world-example-scillating-springs-explained.php) and electronics.

Ex. 3 + 5i is a complex number, here 3 is the real part and 5 is the imaginary part .

 Now we also saw that if a and b were both positive then √ab=√a√b=ab. For a second let’s forget that restriction and do the following.

√−9=√(9)(−1)=√9√−1=3√−1−9=(9)(−1)=9−1=3−1

Now, √−1−1 is not a real number, but if you think about it we can do this for any square root of a negative number. For instance,

√−100=√100√−1=10√−1√−5=√5√−1√−290=√290√−1etc.

The **conjugate** of the complex number a+bi is the complex number a−bi. In other words, it is the original complex number with the sign on the imaginary part changed. Here are some examples of complex numbers and their conjugates.

complex number conjugate 3+12i ,3−12i ; 12−5i , 12+5i ; 1−I , 1+I ; 45i , −45i ; 101 , 101

Notice : that the conjugate of a real number is just itself with no changes.

Now we need to discuss the basic operations for complex numbers. We’ll start with addition and subtraction. The easiest way to think of adding and/or subtracting complex numbers is to think of each complex number as a polynomial and do the addition and subtraction in the same way that we add or subtract polynomials.

Basic Algebraic Operation on Complex Numbers: There are four algebraic operations on complex numbers.

 (i) Addition: If Z1 = a1 + b1 i and Z2 = a2 + b2i,

 then Z1 +Z2 = (a1+ b1i) + (a2 + b2 i) = (a1 + a2) + i(b1 + b2).

(ii) Subtraction: Z2 = (a1+ b1i) – (a2 + b2 i)−Z1 = (a1 – a2) + i(b1 – b2)

 (iii) Multiplication: Z1 Z2 = (a1+ b1i) . (a2 + b2 i) = a1 a2 + b1bi2 + a1b2i + b1a2i = (a1 a2 – b1b2) + i( a1b2 + b1a2)

(iv) Division: z1 z2 = a1 + b1i a2 + b2i Multiply Numerator and denominator by the number a2 – b2 i in order to make the denominator real.

 z1 z2 = a1 + b1i a2 + b2i × a2 – b2i a2 – b2i = (a1a2 + b1b2) + i(b1a2 – a1 b2) a2 2 + b2 2 = + i b1a2 – b1b2 a2 2 + b2 2

Generally result will be expressed in the form a + ib.

Example 1: Add and subtract the numbers 3 + 4i and 2 – 7i.

 Solution: Addition: (3 + 4i) + (2 – 7i) = (3 + 2) + i(4 – 7) = 5 – 3i

 Subtraction: (3 + 4i) – (2 – 7i) = (3 – 2) + i(4 + 7) = 1 + 11i

Example 2: Find the product of the complex numbers: 3 + 4i and 2 – 7i.

 Solution: (3 + 4i) (2 – 7i) = 6 – 21i + 8i – 28i2 = 6 + 28 – 13i = 34 – 13i

 Example 3: Divide 3 + 4i by 2 – 7i.

 Solution: 3 + 4i 2 – 7i = 3 + 4i ×2 – 7i 2 + 7i 2 + 7i = 6 + 28i2 + i(21 + 8) 4 + 49 = –22 + 29i 53 = –22 53 + i 29 53

 Example 4: Express (2 + i) (1 – i) 4 – 3i in the form of a + ib.

Solution: (2 + i)(1 – i) 4 – 3i = (2 + 1) + i(1 – 2) 4 – 3i = 3 – i 4 – 3i = 3 – i 4 – 3i × 4 + 3i 4 + 3i = (12 + 3) + i(9 – 4) 16 + 9 = 15 + i(5) 25 = 15 25 + 5 25 i = 3 5 + 1 5 i

 Example 5: Separate into real and imaginary parts: 1 + 4i 3 + i .

Solution : 1 + 4i 3 + i = 1 + 4i 3 + i × 3 – i 3– i

 Complex Numbers = (3 + 4) + i(12 – 1) 9 + 1 = 7 + 11i 10 = 7 10 + 11 10 i = 7 10 + 11 10 i

 Here, real part = a = 7 10 And imaginary part = b = 11 10.

Extraction of square roots of a complex number:

 Example 6: Extract the square root of the complex numbers 21 – 20i.

Solution: Let a + ib = 21 – 20i

Squaring both sides (a + ib)2 = 21 – 20 i a 2 – b2 + 2abi = 21 – 20 i

Comparing both sides a 2 – b2 = 21…………………….(1)

 2ab = –20………………………(2)

From (2) b = – 10 a Put b in equation (1),

 a 2 – 100 a 2 = 21 a 4 – 21a2 – 100 = 0 (a2 – 25) (a2 + 4) = 0 a 2 = 25 or a2 = –4 a = +5 or a = + –4 = + 2i

 But a is not imaginary, so the real value of a is a = 5 or a = –5 The corresponding value of b is

b = –2 or b = 2

 Hence the square roots of 21 – 20i are: 5 – 2i and –5 + 2i

 Factorization of a complex numbers:

 Example 7: Factorise: a2 + b2

 Solution: We have a2 + b2 = a 2 b2−(− ) =a 2 –( i2 b2 ) , i2 = - 1 = (a)2 – (ib)2 = (a + ib) (a – ib) 7.5

Additive Inverse of a Complex Number:

Let Z = a + ib be a complex number, then the number –a – ib is called the additive inverse of Z . It is denoted by – Z

 Example 8: Find the additive inverse of 2 – 5i

Solution: Let Z =2– 5i Then additive inverse of Z is: – Z – = – (2 – 5i) = – 2 + 5i.

 Multiplicative inverse of a complex number:

 Let a + ib be a complex number, then x + iy is said to be multiplicative inverse of a + ib

 if (x + iy) (a + ib) = 1 Or x + iy = 1 a + ib = 1 ×a + ib a – ib a –ib x + iy = a – ib a 2 + b2 x + iy = a a 2 + b2 – i b a 2 + b2

 So x = a a 2 +b2 y = - b a 2 + b2 ⎝Hence multiplicative inverse of (a, b) is ⎜ ⎛ ⎠ ⎟ ⎞a a 2 –,+ b2 b a 2 + b2

Example 9: Find the multiplicative inverse of 4 + 3i or (4, 3).

 Solution: The multiplicative inverse of 4 + 3i is: 1 4 + 3i Since, 1 4 + 3i = 1 ×4 + 3i 3 – 3i 4 –3i = 4 – 3i 16 + 9 = 4 25 – 3 25 i = ⎝ ⎜ ⎛ ⎠ ⎟ ⎞4 –,25 3 25.

Conjugate of a complex number: Two complex numbers are called the conjugates of each other if their real parts are equal and their imaginary parts differ only in sign. If Z = a + bi, the complex number a – bi is called the conjugate of Z. it is denoted by \_ Z .

i.e., conjugate of a + bi = a – bi.

Modulus of a Complex Number:

The Modulus or the absolute value of the complex number Z = a + ib is denoted by r,

 |Z| or |a + ib| and is given by, r = |Z| = |a + ib| = $\sqrt{a 2 + b2 }$

Thus the modulus |a + ib| is just the distance from the origin to the point (a , b).

Example 11: Find The modulus of (3, – 5) and – 7 – i Solution: |(3, - 5)| = |3 – 5i| = 3 2 + (-5)2 = 9 + 25 = 34 And |-7 – i| = (-7)2 + (-1)2 = 49 + 1 = 50

Theorem: For complex numbers z1 and z2

(i) |z1 . z2| = |z1| |z2| ⎪

(ii) ⎪ ⎪ ⎪ ⎪ ⎪z1 z2 = |z1| |z2|

 (iii) |z1 + z2| < |z1| + |z2|

 (iv) |z1 – z2| > |z1| – |z2|

Polar form of a complex number

 In the fig. 2, we join the point P with the origin, we obtain the line r and .

 Then the numbers or orderθthe angle ) are called the polar coordinatesθpair (r, of the point P to distinguish them from the rectangular co-ordinates (x, y).

We call r the absolute value or , the angle from theθmodulus of z and positive real axis to this line, as the argument or amplitude of z and is denoted = arg z.θby arg z i.e.,

 By use of Pythagorean theorem we have =θCos a r =θ, Sin b r θ , b = r Sin θa = r Cos r = |z | = a 2 +b2 ; r > 0 ; tanθ =$\frac{b}{a}$ and, arg Z= tan –1 ⎝ b/ a ⎞

Therefore, the complex number Z = a + ib = rcos θ + ir Sin θ……………..(1)

This is sometimes written as Z = r Cisθ.

The right hand side of equation (1) is called the Trigonometric or polar form of Z.

 The argZ has any one of an infinite number of real values differing by integral multiple 2kπ, where k = 0, + 1, + 2,………

The values πsatisfying the relation. < θ < is called the principle value of the arg Z,π denoted by Arg. Z. π

Thus arg Z = Arg Z + 2 k

Ex. Express the complex number 1 + i $√$3 in polar form.

Solution: Here, we have a =1 , b = 3 r = a 2 + b2 = 1 +3 = 2 =θand tan b/ a =$√$ 3 = tan-1($√$ 3) = 60

 Hence 1 + i $√$3 = 2 [Cos 60 + i sin 60 ] = 2 Cis 60.

FUNCTIONS OF A COMPLEX VARIABLES

Let S be a set of complex numbers. A function f defined on S is a rule that assigns to each z in S a complex number w.

The number w is called the value of f at z and is denoted by f (z); that is, w = f (z). The set S is called the domain of definition of f .

∗ It must be emphasized that both a domain of definition and a rule are needed in order for a function to be well defined. When the domain of definition is not mentioned, we agree that the largest possible set is to be taken. Also, it is not always convenient to use notation that distinguishes between a given function and its values.

 EXAMPLE 1. If f is defined on the set z$\ne $ 0 by means of the equation w = 1/z, it may be referred to only as the function w = 1/z, or simply the function 1/z.

Suppose that w = u + iv is the value of a function f at z = x + iy, so that u + iv = f (x + iy).

Each of the real numbers u and v depends on the real variables x and y, and it follows that f (z) can be expressed in terms of a pair of real-valued functions of the real variables x and y:

(1) f (z) = u(x, y) + iv(x, y). If the polar coordinates r and θ, instead of x and y, are used, then

u + iv = f (reiθ ) where w = u + iv and z = reiθ .

 In that case, we may write

(2) f (z) = u(r, θ ) + iv(r, θ ).

EXAMPLE 2. If f (z) = z2, then f (x + iy) = (x + iy)2 = x2 − y2 + i2xy.

 Hence u(x, y) = x2 − y2 and v(x, y) = 2xy.

 When polar coordinates are used, f (reiθ ) = (reiθ ) 2 = r2 ei2θ = r2 cos 2θ + ir2 sin 2θ.

 Consequently, u(r, θ ) = r2 cos 2θ and v(r, θ ) = r2 sin 2θ.

EXPONENTIAL FUNCTION

 ez = exeiy  (1)

 (z = x + iy), the two factors ex and eiy being well defined at this time .

 eiy (2) = cos y + i sin y (2)

 is used and y is to be taken in radians.

We see from this definition that ez reduces to the usual exponential function in calculus when y = 0 ; and, following the convention used in calculus, we often write exp z for ez.

 Note that since the positive nth root √n e of e is assigned to ex when x = 1/n (n = 2, 3, . . .), expression (1) tells us that the complex exponential function ez is also √n e when z = 1/n (n = 2, 3, . . .). This is an exception to the convention (Sec. 9) that would ordinarily require us to interpret e1/n as the set of nth roots of e

 Note that definition (1), which can also be written ex+iy = ex eiy, is suggested by the familiar additive property ex1+x2 = ex1 ex2 of the exponential function .

 EXAMPLE. In order to find numbers z = x + iy such that ez  = 1 + i,

 we write equation as ex eiy = √ 2 eiπ/4.

 Then, the equality of two nonzero complex numbers in exponential form, ex = √ 2 and y = π 4 + 2nπ (n = 0, ±1, ±2, . . .).

 Because ln(ex ) = x, it follows that x = ln √ 2 = 1 2 ln 2 and y = 2n + 1 4 π (n = 0, ±1, ±2, . . .); and so z = 1 2 ln 2 + 2n + 1 4

 z = 1 /2 ln 2 + ( 2n + 1/ 4 ) πi , (n = 0, ±1, ±2, . . .).

TRIGONOMETRIC FUNCTIONS

Euler’s formula tells us that

 eix = cos x + i sin x and e−ix = cos x − i sin x for every real number x.

Hence eix − e−ix = 2i sin x and eix + e−ix = 2 cos x.

 That is, sin x = (eix − e−ix )/2i and cos x = (eix + e−ix )/2 .

It is, therefore, natural to define the sine and cosine functions of a complex variable z as follows:

 sin z = (eiz − e−iz )/2i and cos z = (eiz + e−iz)/2 (1) .

 sin(−z) = − sin z, cos(−z) = cos z.

 Also, eiz = cos z + i sin z.

 This is, of course, Euler’s formula when z is real. A variety of identities carry over from trigonometry.

For instance

 sin(z1 + z2) = sin z1 cos z2 + cos z1 sin z2,

 cos(z1 + z2) = cos z1 cos z2 − sin z1 sin z2.

 From these, it follows readily that sin 2z = 2 sin z cos z, cos 2z = cos2 z − sin2 (7) z, sin z + π 2 = cos z, sin z − π 2 (8) = − cos z, and sin2 z + cos2 (9) z = 1.

INVERSE TRIGONOMETRIC FUNTIONS

Inverses of the trigonometric and hyperbolic functions can be described in terms of logarithms. In order to define the inverse sine function sin−1 z, we write w = sin−1 z when z = sin w.

That is, w = sin−1 z when z = eiw − e−iw 2i .

 If we put this equation in the form (eiw) 2 − 2iz(eiw) − 1 = 0.