SIMPLEX METHOD

* 1. Introduction

The two variable problem of the LPP can be solved by the graphical method, but it is very complicated to solve the three or more variable problem by using the graphical method. In such cases, a simplex and most widely used simplex method is adopted, which was developed by G. B, Dantzig in 1947. The simplex method provides an algorithm which is based on the fundamental theorem of linear programming. See the ﬁgure below;

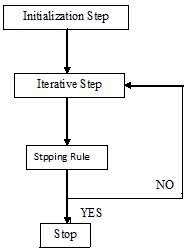


Figure 4.1: Structure of an algorithms

* 1. Standard Form of an LPP

We have to convert the LPP into the standard form of LPP before the use of simplex method. The standard form of the LPP should have the following char- acteristics;

* + 1. All the constraints should be expressed as equations by adding slack or sur- plus and / or artiﬁcial variables.
    2. The right hand side of each constraints should be made non negative if it is not, this should be done by multiplying both sides of the resulting constraints by -1.
    3. The objective function should be of the maximization type

The general standard form of the LPP is expressed as follows;

*Optimize Z* = *c*1*x*1 + *c*2*x*2 + *...* + *cnxn* + 0*S*1 + 0*S*2 + *...* + 0*Sm* subjected to the constraints

*a*11*x*1 +*a*12*x*2 +*...* +*a*1*jxj* + *... a*1*nxn* + *S*1(≤=≥)*b*1 *a*21*x*1 +*a*22*x*2 +*...* +*a*2*jxj* + *... a*2*nxn* + *S*2(≤=≥)*b*2

.**. .**. **.**. **.**. **.**.

*ai*1*x*1 +*ai*2*x*2 +*...* +*aijxj* + *... ainxn* + *Sn*(≤=≥)*bi*

.**. .**. **.**. **.**. **.**.

*am*1*x*1 +*am*2*x*2 +*...* +*amixj* + *... amnxn* + *Sm*(≤=≥)*bn*

and non negativity constraints

*x*1*, x*2*,..., xn, S*1*, S*2*,..., Sm* ≥ 0

Note:

* + - 1. A slack variable represents unused resource, either in the form of time on a machine, labour hours, money, warehouse space or any number of such resources in various business problems. Since these variables yield no proﬁt, therefore such variables are added to the original objective function with zero coeﬃcients. Slack variables are also deﬁned as the non-negative variables

which are added in the LHS of the constraints to convert the inequality j ≤j

into an equation.

* + - 1. A surplus variable represents amount by which solution values exceed a re- source. These variables are also called negative slack variables. Surplus variables, like slack variable carry a zero coeﬃcient in the objective func- tion. Surplus variables which are removed from the LHS of the constraints to convert the inequality j ≥j into an equation.
      2. Artiﬁcial variables are also deﬁned as the non-negative variables which are added in the LHS of the constraints to convert equality into the standard form of simplex.
  1. The Simplex Method
     1. Maximization Case

The steps of the simplex algorithm to obtain an optimal solution(if it exists) to the LPP are as follows. But before you start step 1, ﬁrst formulate the mathematical model of the given LPP.

Step 1: Express the Problem in Standard Form

* + - * Check whether the objective function of the formulated LPP is of max- imization or minimization. If it is of minimization, then convert it into one of maximization by using the following relationship.

*Minimize Z* = −*Maximize Z*∗ ∗

*where Z* = −*Z*

* + - * Check whether all the *bi*(*i* = 1*,* 2*,..., m*) values are positive. If any one of them is negative, then multiply the corresponding constraint by -1 in order to make *bi* ≥ 0. In doing so, remember to change a ≤ type constraint to a ≥ type constraint, and vice-versa.
      * Replace each unrestricted variable with the diﬀerence of two non-negative variables; replace each non-positive variable with a new non-negative variable whose value is the negative of the original variable.
      * After that express the problem in standard form by introducing slack, surplus and/or artiﬁcial variables, to convert the inequalities into equa- tions.

Step 2:Find the Initial Basic Solution

* + - * In the simplex method, a start is made with a basic feasible solution, which we shall get by assuming that the objective function value Z=0. This will be so when decision variables *x*1*, x*2*,..., xn* each equal to zero. These variables are called non-basic variables.
      * Substituting *x*1 = *x*2 = *...* = *xn* = 0 in constraint equations we get *S*1 = *b*1*, S*2 = *b*2 *... Sm* = *bm* which is called initial basic feasible solution. Not that *Z* = 0 for this solution.
      * Variables *S*1*, S*2*,..., Sm* are called basic variable (BV).
      * The problem in the standard form and the solution obtained above are now expressed in the form of table, called simplex tableau.

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| *Cj* −→ | | | *C*1 | *C*2 | ... | *Cn* | 0 | . . . 0 |  |
| *CB* | B | *b*(=  *xB*) | *x*1 | *x*2 | ... | *xn* | *S*1 | ... *Sn* | Min. |
| *CB*1 | *S*1 | *xB*1 = | *a*11 | *a*12 | ... | *a*1*n* | 1 | . . . 0 |  |
|  |  | *b*1 |  |  |  |  |  |  |
| *CB*2 | *S*2 | *xB*2 = | *a*21 | *a*22 | ... | *a*2*n* | 0 | . . . 0 |
|  |  | *b*2 |  |  |  |  |  |  |
| **.** | **.** | **.** | **.** | **.** | **.** | **.** | **.** | **.** |
| *CBm* | *Sm* | *xBm* = | *am*1 | *am*2 | ... | *amn* | 0 | . . . 1 |
|  |  | *bm* |  |  |  |  |  |  |
| *Z* =  Σ*CBmxBm* | | *Zj* =  Σ*CBmxj* | 0 | 0 | ... | 0 | 0 | ...0 |  |
| ... | ... | *Cj* −*Zj* | *C*1 − *Z*1 | *C*2 − *Z*2 | ... | *Cn* − *Zn* | 0 | . . . 0 |  |

Ratio

Where;

* + - * *Cj*: Objective row (Coeﬃcient of variable in objective function) it remain unchanged during succeeding table.
      * *CBm*: Objective column (Coeﬃcient of current basic variable in objective function)
      * *Sm*: Basic variable in basic. Initially basic variables are slack variables.
      * *xBm*: Values of basic variables column when *x*1 = *x*2 = *...* = *xn* = 0.
      * Body Matrix: Coeﬃcient of decision (non-basic) variables in constraints set (*aij*).
      * Identity Matrix: Coeﬃcient of slack variables in the table.
      * *Z*: It presents the proﬁt or loss *Z* = Σ(*CBmxBm*) .
      * *Cj* − *Zj*: It presents the index row.

Step 3: Perform Optimality Test

* + - * Calculate the elements of index row (*Cj* −*Zj*), if all the elements in index row are negative then, current solution is optimum basic solution, if not then go for next step.

Step 4: Iterate Towards an Optimal Solution

* + - * If step 3 does not holds, then select a variable that has the largest *Cj* −*Zj* value to enter into the new solution. That is *Ck* − *Zk* = *Max* [(*Cj* − *Zj*); *Cj* − *Zj* ≥ 0]. The column to be entered is called the key or pivot column. Such variable indicates the largest per unit improvement in the current solution.
      * Identify key or pivot row, corresponding to smallest non-negative ratio

found by dividing the values. That is *xBr* = *Min xBi* ; *a >* 0. It should

*arj*

*arj rj*

be noted that division by negative or zero element is not permitted.

* + - * Identify key element, the non-zero positive element at the intersection of key column and key row, circle the key element.
      * Construct new simplex table by calculating the new values for the key row by dividing every element of the key row by the key element, if the key element is not 1, otherwise the key row remain unchanged.
      * The new values of the elements in the remaining rows for the new simplex table can be obtained by performing elementary row operations on all rows so that all elements except the key element in the key column are zero. We use the following formula for the new row other than key row;

*NewRowNo.* = (*No.inOldRow*) − (*AssociateNo.inKeyRow*) ×

*CorrespondingNo.inKeyColumn KeyElement*

. Σ

Step 5: Repeat the Procedure

* + - * Go to step 3 and repeat the procedure until either an optimal solution is reached or there is an indication of unbounded solution. We will see later on, how you can determine the unbounded solution for the given LPP.

Example 1: Solve the following LPP by using simplex method;

*Max Z* = 6*x*1 + 4*x*2

Subject to

*x*1 + 2*x*2 ≤ 720 2*x*1 + *x*2 ≤ 780

*x*1 ≤ 320

Solution:

Step 1: Convert the Following LPP into Standard Form

*Max Z* = 6*x*1 + 4*x*2 + 0*S*1 + 0*S*2 + 0*S*3

Subject to

*x*1 + 2*x*2 + *S*1 = 720 2*x*1 + *x*2 + *S*2 = 780

*x*1 + *S*3 = 320

Step 2: Initial Basic Feasible Solution

*x*1 = 0 *and x*2 = 0 in the above equation then we have *S*1 = 720*, S*2 = 780 *and S*3 = 320

Step 3: Perform the Optimality Test

Since all *Cj* − *Zj* ≥ 0(*j* = 1*,* 2), the current solution is not optimal. Variable *x*1 is chosen to enter into the basis as *C*1 − *Z*1 = 6 is the largest positive number in the *x*1 column, where all elements are positive. This means that for every unit of variable *x*1, the objective function will increase in value by 6. The *x*1 column is the key column.

Table 4.1: Initial Solution

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  |  | *C*j −→ | 6 | 4 | 0 | 0 | 0 |  | |
| *C*B | B | *b*(= *x*B) | *x*1 | *x*2 | *S*1 | *S*2 | *S*3 | Min.Ratio | |
| 0 | *S*1 | 720 | 1 | 2 | 1 | 0 | 0 | 720 | = 720  = 390  = 320 → |
| 1 |
| 0 | *S*2 | 780 | 2 | 1 | 0 | 1 | 0 | 780 |
| 2 |
| 0 | *S*3 | 320 | 1 | 0 | 0 | 0 | 1 | 320 |
| 1 |
| *Z* = 0 |  | *Z*j = | 0 | 0 | 0 | 0 | 0 |  | |
|  |  | *C*j − *Z*j | 6  ↑ | 4 | 0 | 0 | 0 |  | |

Step 4: Determine the Variable to Leave the Basis

The variable to leave the basis is determined by dividing the value in the *xB*- (constant) column by their corresponding elements in the key column as shown in Table 4.1. Since the exchange ratio, 320 is minimum in row 3, the basic variable *S*3 is chosen to leave the solution basis.

Iteration 1:Since the key element enclosed in the circle in Table 4.1 is 1, this row remain unchanged. The new values of the elements in the remaining rows for the new Table is obtained by performing the following elementary row operations on all rows so that all elements except the key element 1 in the key column are zero.

*R*3(*old*)

*R*3(*new*) →

1(*keyelement*)

= (320*,* 1*,* 0*,* 0*,* 0*,* 1)

*R*2(*new*) → *R*2(*old*) − 2*R*3(*new*)

*R*2(*new*) → (780*,* 2*,* 1*,* 0*,* 1*,* 0) − 2(320*,* 1*,* 0*,* 0*,* 0*,* 1) = (140*,* 0*,* 1*,* 0*,* 1*,* −2)

*R*1(*new*) → *R*1(*old*) − 1*R*3(*new*)

*R*1(*new*) → (720*,* 1*,* 2*,* 1*,* 0*,* 0) − 1(320*,* 1*,* 0*,* 0*,* 0*,* 1) = (400*,* 0*,* 2*,* 1*,* 0*,* −1)

Then, the new improved solution is given in 4.2 below;

An improved basic feasible solution can be read from Table 4.2 as: *x*1 = 320*, S*2 = 140*, S*3 = 400 *and x*2 = 0. The improved value of objective function is Z=1920.

Once again, calculate values of *Cj* − *Zj* in the same manner as we have done to get the improved solution in Table 4.2 to see whether the solution is optimal or not. Since *C*2 − *Z*2 *>* 0, the current solution is not optimal.

Table 4.2: Improved Solution

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  | *C*j −→ | 6 4 0 0 0 |  |
| *C*B | B | *b*(= *x*B) | *x*1 *x*2 *S*1 *S*2 *S*3 | Min.Ratio |
| 0  0  6 | *S*1 *S*2  *x*1 | 400  140  320 | 0 2 1 0 -1  0 1 0 1 -2  1 0 0 0 1 | 400  = 200  2  140  = 140 →  1 |
| *Z* = 1920 |  | *Z*j = | 6 0 0 0 6 |  |
|  |  | *C*j − *Z*j | 0 4 0 0 -6  ↑ |  |

Iteration 2:Repeats steps 3 to 4. Table 4.3 is obtained by performing following row operations to enter *x*2 into the basis and to drive out *S*2 from the basis.

*R*2(*old*)

*R*2(*new*) →

1(*key element*)

= (140*,* 0*,* 1*,* 0*,* 1*,* −2)

*R*1(*new*) → *R*1(*old*) − 2*R*2(*new*)

*R*1(*new*) → (400*,* 0*,* 2*,* 1*,* 0*,* −1) − 2(140*,* 0*,* 1*,* 0*,* 1*,* −2) = (120*,* 0*,* 0*,* 1*,* −2*,* 3)

*R*3(*new*) → *R*3(*old*) − 0*R*2(*new*)

*R*3(*new*) → (320*,* 1*,* 0*,* 0*,* 0*,* 1) − 0(140*,* 0*,* 1*,* 0*,* 1*,* −2) = (320*,* 1*,* 0*,* 0*,* 0*,* 1)

Then, the improved solution for iteration 2 is given in Table 4.3 below;

Table 4.3: Improved Solution

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  | *C*j −→ | 6 4 0 0 0 |  |
| *C*B | B | *b*(= *x*B) | *x*1 *x*2 *S*1 *S*2 *S*3 | Min.Ratio |
| 0  4  6 | *S*1  *x*2 *x*1 | 120  140  320 | 0 0 1 -2 3  0 1 0 1 -2  1 0 0 0 1 | 120  = 40 →  3  320  = 320  1 |
| *Z* = 2480 |  | *Z*j = | 6 4 0 4 -2 |  |
|  |  | *C*j − *Z*j | 0 0 0 -4 2  ↑ |  |

Iteration 3:Repeats steps 3 to 4. Table 4.4 is obtained by performing following

row operations to enter *S*3 into the basis and to drive out *S*1 from the basis.

*R*1(*old*)

*R*1(*new*) →

3(*key element*)

= (40*,* 0*,* 0*,* 1*/*3*,* −2*/*3*,* 1)

*R*2(*new*) → *R*2(*old*)+ 2*R*1(*new*)

*R*2(*new*) → (140*,* 0*,* 1*,* 0*,* 1*,* −2) + 2(40*,* 0*,* 0*,* 1*/*3*,* −2*/*3*,* 1) = (220*,* 0*,* 1*,* 2*/*3*,* −1*/*3*,* 0)

*R*3(*new*) → *R*3(*old*) − 1*R*1(*new*)

*R*3(*new*) → (320*,* 1*,* 0*,* 0*,* 0*,* 1) − 1(40*,* 0*,* 0*,* 1*/*3*,* −2*/*3*,* 1) = (280*,* 1*,* 0*,* −1*/*3*,* 2*/*3*,* 0)

Then, the improved solution for iteration 2 is given in Table 4.4 below;

Table 4.4: Optimal Solution

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  | *C*j −→ | 6 4 0 0 0 |  |
| *C*B | B | *b*(= *x*B) | *x*1 *x*2 *S*1 *S*2 *S*3 | Min.Ratio |
| 0  4  6 | *S*3  *x*2 *x*1 | 40  220  280 | 0 0 1/3 -2/3 1  0 1 2/3 -1/3 0  1 0 -1/3 2/3 0 |  |
| *Z* = 2560 |  | *Z*j = | 6 4 2/3 8/3 0 |  |
|  |  | *C*j − *Z*j | 0 0 -2/3 -8/3 0 |  |

Since all *Cj* − *Zj* ≤ 0 corresponding to non - basic variables columns, the current solution cannot be improved further. This means that the current basic feasible solution is also the optimal solution. Thus, *x*1 = 280*, x*2 = 220 and the value of objective function is Z=2560. Example 2: Use the simplex method to solve following LP problem.

*Max Z* = 6*x*1 + 17*x*2 + 10*x*3

Subject to

*x*1 + *x*2 + 4*x*3 ≤ 2000 2*x*1 + *x*2 + *x*3 ≤ 3600 *x*1 + 2*x*2 + 2*x*3 ≤ 2400

*x*1 ≤ 30

and

*x*1*, x*2*, x*3 ≥ 0

Solution:

Convert the Following LPP into Standard Form

*Max Z* = 6*x*1 + 17*x*2 + 10*x*3 + 0*S*1 + 0*S*2 + 0*S*3 + 0*S*4

Subject to

*x*1 + *x*2 + 4*x*3 + *S*1 = 2000 2*x*1 + *x*2 + *x*3 + *S*2 = 3600 *x*1 + 2*x*2 + 2*x*3 + *S*3 = 2400

*x*1 + *S*4 = 30

and

*x*1*, x*2*, x*3*, S*1*, S*2*, S*3*, S*4 ≥ 0

Initial Basic Feasible Solution

An initial basic feasible solution is obtained by setting *x*1 = *x*2 = *x*3 = 0. Thus, the initial solution is: *S*1 = 2000*, S*2 = 3600*, S*3 = 2400*, S*4 = 30 and Max Z = 0. The solution can also be read from the initial simplex Table 4.5.

Table 4.5: Initial Solution

|  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  |  | *C*j −→ | 6 | 17 | 10 | 0 | 0 | 0 | 0 |  | |
| *C*B | B | *b*(= *x*B) | *x*1 | *x*2 | *x*3 | *S*1 | *S*2 | *S*3 | *S*4 | Min.Ratio | |
| 0 | *S*1 | 2000 | 1 | 1 | 4 | 1 | 0 | 0 | 0 | 2000  1  3600  1  2400  2  − | = 2000 |
| 0 | *S*2 | 3600 | 2 | 1 | 1 | 0 | 1 | 0 | 0 | = 3600 |
| 0 | *S*3 | 2400 | 1 | 2 | 2 | 0 | 0 | 1 | 0 | = 1200 → |
| 0 | *S*4 | 30 | 1 | 0 | 0 | 0 | 0 | 0 | 1 |  |
| *Z* = 0 |  | *Z*j = | 0 | 0 | 0 | 0 | 0 | 0 | 0 |  | |
|  |  | *C*j − *Z*j | 16 | 17  ↑ | 10 | 0 | 0 | 0 | 0 |  | |

Perform the Optimality Test

Since all *Cj* − *Zj* ≥ 0, the current solution is not optimal. Variable *x*2 is chosen to enter into the basis as *C*2 − *Z*2 = 17 is the largest positive number in the *x*2 column. We apply the following row operations to get a new improved solution and removing *S*3 from the basis.

*R*3(*old*)

*R*3(*new*) −→

2(*key element*)

= (1200*,* 1*/*2*,* 0*,* 0*,* 0*,* 1*,* −1*/*2*,* 0)

*R*1(*new*) −→ *R*1(*old*) − *R*3(*new*) = (800*,* 1*/*2*,* 0*,* 3*,* 1*,* 0*,* −1*/*2*,* 0)

*R*2(*new*) −→ *R*2(*old*) − *R*3(*new*) = (2400*,* 3*/*2*,* 0*,* 0*,* 0*,* 1*,* −1*/*2*,* 0)

*R*4(*new*) −→ *R*4(*old*) = (30*,* 1*,* 0*,* 0*,* 0*,* 0*,* 0*,* 1)

The new solution is shown in Table 4.6

Table 4.6: Improved Solution

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  | *C*j −→ | 6 17 10 0 0 0 0 |  |
| *C*B | B | *b*(= *x*B) | *x*1 *x*2 *x*3 *S*1 *S*2 *S*3 *S*4 | Min.Ratio |
| 0  0  17  0 | *S*1 *S*2  *x*2  *S*4 | 800  2400  1200  30 | 1/2 0 3 1 0 -1/2 0  3/2 0 0 0 1 -1/2 0  1/2 1 1 0 0 1/2 0  1 0 0 0 0 0 1 | 800  = 1600  1*/*2  2400  = 1600  3*/*2  1200  = 2400  1*/*2  30  = 30 →  1 |
| *Z* = 20*,* 000 |  | *Z*j = | 17/2 17 17 0 0 17/2 0 |  |
|  |  | *C*j − *Z*j | 15/2 0 -7 0 0 -17/2 0  ↑ |  |

The solution shown in Table 4.6 is not optimal because *C*1 −*Z*1 = 15*/*2 which is positive in *x*1 column. Thus, applying the following row operations to get new improved solution by entering variable *x*1 into the basis and removing the variable *S*4 from the basis.

*R*4(*new*) →

*R*4(*old*) 1(*key element*)

= (30*,* 1*,* 0*,* 0*,* 0*,* 0*,* 0*,* 1)

*R*1(*new*) → *R*1(*old*) − (1*/*2)*R*4(*new*) = (785*,* 0*,* 0*,* 3*,* 1*,* 0*,* −1*/*2*,* −1*/*2)

*R*2(*new*) → *R*2(*old*) − (3*/*2)*R*4(*new*) = (2355*,* 0*,* 0*,* 0*,* 0*,* 1*,* −1*/*2*,* −3*/*2)

*R*3(*new*) → *R*3(*old*) − (1*/*2)*R*4(*new*) = (1185*,* 0*,* 1*,* 1*,* 0*,* 0*,* 1*/*2*,* −1*/*2)

Table 4.7: Optimal Solution

|  |  |  |  |
| --- | --- | --- | --- |
|  |  | *C*j −→ | 6 17 10 0 0 0 0 |
| *C*B | B | *b*(= *x*B) | *x*1 *x*2 *x*3 *S*1 *S*2 *S*3 *S*4 |
| 0  0  17  16 | *S*1 *S*2  *x*2  *x*1 | 785  2355  1185  30 | 0 0 3 1 0 -1/2 -1/2  0 0 0 0 1 -1/2 -3/2  0 1 1 0 0 1/2 -1/2  1 0 0 0 0 0 1 |
| *Z* = 20*,* 625 |  | *Z*j = | 16 17 17 0 0 17/2 15/2 |
|  |  | *C*j − *Z*j | 0 0 -7 0 0 -17/2 -15/2 |

Then, the improved solution for this iteration is given in Table 4.7 below;

Since all *Cj* − *Zj* ≤ 0 corresponding to non - basic variables columns, the current solution cannot be improved further. This means that the current basic feasible solution is also the optimal solution. Thus, *x*1 = 30*, x*2 = 1*,* 185 and *x*3 = 0 to obtain the maximum value of Z=20,625.

Activities

1. A manufacturer of leather belts makes three types of belts A,B and C which are processed on three machines *M*1*, M*2 and *M*3. Belts A requires 2 hours on machines *M*1 and 3 hours on machine *M*2 and 2 hours on machine *M*3. Belts B requires 3 hours on machine *M*1, 2 hours on machine *M*2 and 2 hours on machine *M*3 and Belt C requires 5 hours on machine *M*2 and 4 hours on machine *M*3. There are 8 hours of time per day available on machine *M*1, 10 hours of time per day available on machine *M*2 and 15 hours of time per day available on machine *M*3. The proﬁt gained from belt A is 3 USD per unit, from belt B is 5 USD per unit, from belt C is 4 USD per unit. What should be the daily production of each type of belt so that the proﬁt is maximum?
2. A farmers has 1,000 acres of land on which he can grow corn, wheat or soyabean. Each acre of corn costs 100 USD for preparation, requires 7 men- days of work and yields a proﬁt of 30 USD. An acre of wheat costs USD 120 to prepare, requires 10 men-days of work and yields a proﬁt of 40 USD. An acre of soyabean costs 70 USD to prepare, requires 8 men-days of work and yields a proﬁt of 20 USD. If the farmer has 1,000,000 for preparation and can count on 8,000 men-days of work, determine how many acres should be allocated to each crop to maximize proﬁts?
   * 1. Minimization Case

In certain cases it is diﬃcult to obtain an initial basic feasible solution, such case arise;

When the constraints are of the ≤ type

*n*

*aijxj* ≤ *bi, xj* ≥ 0

Σ

*j*=1

but some right-hand side constants are negative (*bi <* 0). In this case, after adding the non-negative slack variable *Si*, the initial solution so obtained will be *Si* = −*bi* for some *i*. It is not the feasible solution because it violates the non-negativity condition of slack variables.

When the constraints are of ≥ type

*n*

*aijxj* ≥ *bi, xj* ≥ 0

Σ

*j*=1

In this case to convert the inequalities into equation form, add surplus (neg- ative slack) variables

*n*

*aijxj* − *Si* = *bi, xj, Si* ≥ 0

Σ

*j*=1

Letting *xj* = 0, we get an initial solution −*Si* = *bi* or *Si* = −*bi*. It is also not a feasible solution as it violates the non-negativity condition of surplus variables.

In this case we add artiﬁcial variables *Ai* to get an initial basic feasible solution. The resulting system of equations then becomes;

*n*

*aijxj* − *Si* + *Ai* = *bi, xj, Si, Ai* ≥ 0*,i* = 1*,* 2*,* 3*, ..., m*

Σ

*j*=1

and has m equations and (*n* + *m* + *m*) variables (i.e n-decision variables, m artiﬁcial variables and m surplus variables).

To get back to the original problem, artiﬁcial variables must be dropped out of the optimal solution. There are two methods for eliminating these variables from the solution

1. Two - Phase Method
2. Big-M Method or Method of Penalties.
   * 1. The Two-Phase Method

In the ﬁrst phase of this method the sum of all artiﬁcial variables is minimized subject to the given constraints to get a basic feasible solution of the LPP.

The second phase minimizes the original objective function starting with the basic feasible solutio obtained at the end of the ﬁrst phase. The steps of the algorithm is given bellow;

Phase I:

1. (a) If all the constraints in the given LPP are ≤ type then go to Phase

II. Otherwise, add some surplus and artiﬁcial variables to get equality con- straints.

(b) If the given LPP is of minimization then convert to maximization.

1. Assign zero coeﬃcients to each of the decision variables *xj* and to the surplus variables and assign -1 coeﬃcient to each of the artiﬁcial variables. This yields the following auxiliary LPP;

subject to

*Max Z*∗

*m*

= (−1)*Ai*

Σ

*i*=1

*n*

*aijxj* + *Ai* = *bi, xj, Ai* ≥ 0*,i* = 1*,* 2*,* 3*, ..., m*

Σ

*j*=1

1. Apply the simplex algorithm to solve this auxiliary LPP. The following three cases may arise at optimality;

Max *Z* = 0 and atleast one artiﬁcial variable is present in the basis with positive value. Then no feasible solution exists for the original LPP.

∗

Max *Z* = 0 and no artiﬁcial variable is present in the basis. Then the basis consists of only decision variables *x*j *s* and hence we may move to Phase II to obtain an optimal basic feasible solution on the original LPP.

∗

*j*

Max *Z* = 0 and atleast one artiﬁcial variable is present in the basis at zero value. Then a feasible solution to the above LPP is also a feasible solution to the original LPP. Now we may proceed direct to Phase II.

∗

Phase II:

Assign actual coeﬃcients to the variables in the objective function and zero to the artiﬁcial variables which appear at zero value in the basis at the end of Phase I. Then apply the usual simplex algorithm to the modiﬁed simplex table to get optimal solution to the original problem. Artiﬁcial variables which do not appear in the basis may be removed.

Example 1:Solve the following LP model using Two-Phase Method;

*Max Z* = 5*x*1 − 4*x*2 + 3*x*3

subject to

and

Solution:

2*x*1 + *x*2 − 6*x*3 = 20

6*x*1 + 5*x*2 + 10*x*3 ≤ 76

8*x*1 − 3*x*2 + 6*x*3 ≤ 50

*x*1*, x*2*, x*3 ≥ 0

After adding surplus variables *S*1 and *S*2 and artiﬁcial variable *A*1 the prob- lem becomes;

subject to

and

Phase I:

*Max Z* = 5*x*1 − 4*x*2 + 3*x*3

2*x*1 + *x*2 − 6*x*3 + *A*1 = 20

6*x*1 + 5*x*2 + 10*x*3 + *S*1 = 76

8*x*1 − 3*x*2 + 6*x*3 + *S*2 = 50

*x*1*, x*2*, x*3*, S*1*, S*2*, A*1 ≥ 0

Construction of Auxiliary LP model

∗

*Max Z* = −*A*1

subject to

2*x*1 + *x*2 − 6*x*3 + *A*1 = 20

6*x*1 + 5*x*2 + 10*x*3 + *S*1 = 76

8*x*1 − 3*x*2 + 6*x*3 + *S*2 = 50

and

*x*1*, x*2*, x*3*, S*1*, S*2*, A*1 ≥ 0

Solution of an Auxiliary LP model

Table 4.8: Initial Solution

|  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  |  | *C*j | −→ | 0 | 0 | 0 | -1 | 0 | 0 |  | |
| *C*B | B | *b*(= *x*B) | | *x*1 | *x*2 | *x*3 | *A*1 | *S*1 | *S*2 | Min.Ratio | |
| -1 | *A*1 | 20 | | 2 | 1 | -6 | 1 | 0 | 0 | 20 | = 10  = 12*.*66  = 6*.*25 → |
| 2 |
| 0 | *S*1 | 76 | | 6 | 5 | 10 | 0 | 1 | 0 | 76 |
| 6 |
| 0 | *S*2 | 50 | | 8 | -3 | 6 | 0 | 0 | 1 | 50 |
| 8 |
| *Z* = −20 |  | *Z*j = | | -2 | -1 | 6 | -1 | 0 | 0 |  | |
|  |  | *C*j − *Z*j | | 2  ↑ | 1 | -6 | 0 | 0 | 0 |  | |

Slack variable *S*2 is removed from the basis since it has minimum ratio and variable *x*1 is entering the basis since it has highest positive value into *Cj* −*Zj* row.

Iteration 1: The improved solution is obtained by performing the following elementary row operations.

*R*3(*new*) →

*R*3(*old*) 8(*key element*)

= (25*/*4*,* 1*,* −3*/*8*,* 3*/*4*,* 0*,* 0*,* 1*/*8)

*R*1(*new*) → *R*1(*old*) − (2)*R*3(*new*) = (15*/*2*,* 0*,* 7*/*4*,* −15*/*2*,* 1*,* 0*,* −1*/*4)

*R*2(*new*) → *R*2(*old*) − (6)*R*3(*new*) = (77*/*2*,* 0*,* 29*/*4*,* 11*/*2*,* 0*,* 1*,* −3*/*4)

The improved solution is given in Table 4.9

Iteration 2: To remove *A*1 from the solution shown in Table 4.9 above, enter *x*2 in the basis by applying the following elementary row operations.

*R*1(*old*)

*R*1(*new*) →

7*/*4(*key element*)

= (30*/*7*,* 0*,* 1*,* −30*/*7*,* 4*/*7*,* 0*,* −1*/*7)

*R*2(*new*) → *R*2(*old*) − (29*/*4)*R*1(*new*) = (52*/*7*,* 0*,* 1*,* 256*/*7*,* 1*,* 2*/*7)

*R*3(*new*) → *R*3(*old*) − (−3*/*8)*R*1(*new*) = (55*/*7*,* 1*,* 0*,* −6*/*7*,* 3*/*4*,* 0*,* 1*/*14)

The improved solution is given in Table 4.10

Table 4.9: Improved Solution

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  | *C*j −→ | 0 0 0 -1 0 0 |  |
| *C*B | B | *b*(= *x*B) | *x*1 *x*2 *x*3 *A*1 *S*1 *S*2 | Min.Ratio |
| -1  0  0 | *A*1 *S*1  *x*1 | 15/2  77/2  25/4 | 0 7/4 -15/2 1 0 -1/4  0 29/4 11/2 0 1 -3/4  1 -3/8 3/4 0 0 1/8 | 15*/*2  = 30*/*7 →  7*/*4  77*/*2  = 154*/*29  29*/*4  - |
| *Z* = −15*/*2 |  | *Z*j = | 0 -7/4 15/2 -1 0 1/4 |  |
|  |  | *C*j − *Z*j | 0 7/4 -15/2 0 0 -1/4  ↑ |  |

Table 4.10: Improved Solution

|  |  |  |  |
| --- | --- | --- | --- |
|  |  | *C*j −→ | 0 0 0 -1 0 0 |
| *C*B | B | *b*(= *x*B) | *x*1 *x*2 *x*3 *A*1 *S*1 *S*2 |
| 0  0  0 | *x*2 *S*1  *x*1 | 30/7  52/7  55/7 | 0 1 -30/7 4/7 0 -1/7  0 1 256/7 -29/7 1 2/7  1 0 -6/7 3/4 0 1/14 |
| *Z* = 0 |  | *Z*j = | 0 0 0 0 0 0 |
|  |  | *C*j − *Z*j | 0 0 0 -1 0 0 |

Since all *Cj* − *Zj* ≤ 0 an optimal solution to the auxiliary LP model has been obtained and Max Z=0 with no artiﬁcial variable in the basis.

item However, this solution may or may not be the basic feasible solution to the original LPP. Thus, go to Phase II to get an optimal solution to our original LPP.

Phase II

The modiﬁed simplex table from Table 4.10 is as follows;

Since all *Cj* − *Zj* ≤ 0 for all non-basic variables, the current basic feasible solution is also optimal. Hence, an optimum feasible solution to the given LPP is *x*1 = 55*/*7*, x*2 = 30*/*7*, x*3 = 0*, S*1 = 52*/*7*, S*2 = 0*, S*3 = 0 and Max. *Z* = 155*/*7.

Example 2: Solve the following LPP by using two-phase method;

*Min Z* = *x*1 − 2*x*2 − 3*x*3

Table 4.11: Modiﬁed Simplex Table

|  |  |  |  |
| --- | --- | --- | --- |
|  |  | *C*j −→ | 5 -4 0 0 0 |
| *C*B | B | *b*(= *x*B) | *x*1 *x*2 *x*3 *S*1 *S*2 |
| -4  0  5 | *x*2 *S*1  *x*1 | 30/7  52/7  55/7 | 0 1 -30/7 0 -1/7  0 1 256/7 1 2/7  1 0 -6/7 0 1/14 |
| *Z* = 155*/*7 |  | *Z*j = | 5 -4 90/7 0 13/14 |
|  |  | *C*j − *Z*j | 0 0 -69/7 0 -13/14 |

subject to

and

Solution:

−2*x*1 + 3*x*2 + 3*x*3 = 2 2*x*1 + 3*x*2 + 4*x*3 = 1

*x*1*, x*2*, x*3 ≥ 0

After converting the objective function into maximization and adding arti- ﬁcial variables *A*1 and *A*2 in the constraints of the given LPP, the problem becomes;

subject to

and

Phase I:

∗

*Max Z* = −*x*1 + 2*x*2 + 3*x*3

−2*x*1 + 3*x*2 + 3*x*3 + *A*1 = 2 2*x*1 + 3*x*2 + 4*x*3 + *A*2 = 1

∗

*x*1*, x*2*, x*3*, A*1*, A*2 ≥ 0 *where Z* = −*Z*

Construction of Auxiliary LP model

∗

*Max Z* = −*A*1 − *A*2

subject to

−2*x*1 + 3*x*2 + 3*x*3 + *A*1 = 2 2*x*1 + 3*x*2 + 4*x*3 + *A*2 = 1

and

*x*1*, x*2*, x*3*, A*1*, A*2 ≥ 0

The initial solution of an Auxiliary LPP is given bellow;

Table 4.12: Initial Solution

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  |  | *C*j | −→ | 0 | 0 | 0 | -1 | -1 |  |
| *C*B | B | *b*(= *x*B) | | *x*1 | *x*2 | *x*3 | *A*1 | *A*1 | Min.Ratio |
| -1 | *A*1 | 2 | | -2 | 1 | 3 | 1 | 0 | 2  = 0*.*67  1 3  = 0*.*25 →  4 |
| -1 | *A*2 | 1 | | 2 | 3 | 4 | 0 | 1 |
| *Z*∗ = −3 |  | *Z*j = | | 0 | -4 | -7 | -1 | -1 |  |
|  |  | *C*j − *Z*j | | 0 | 4 | 7  ↑ | 0 | 0 |  |

Artiﬁcial variable *A*2 is removed from the basis since it has minimum ratio and variable *x*3 is entering the basis since it has highest positive value into *Cj* − *Zj* row.

Iteration 1: The improved solution is obtained by performing the following

elementary row operations.

*R*2(*new*) →

*R*2(*old*) 4(*key element*)

= (1*/*4*,* 1*/*2*,* 3*/*4*,* 1*,* 0)

*R*1(*new*) → *R*1(*old*) − (3)*R*2(*new*) = (5*/*4*,* −7*/*2*,* −5*/*4*,* 0*,* 1)

The improved solution so obtained is given in Table 4.13. Since in Table 4.13, *Cj* − *Zj* ≤ 0 corresponds to non-basic variables, the optimal solution is *x*1 = 0*, x*2 = 0*, x*3 = 1*/*4*, A*1 = 5*/*4 and *A*2 = 0 with Max *Z*∗ = −5*/*4. But at the same time, the value of *Z*∗ *<* 0 and the artiﬁcial variable *A*1 appears in the basis with positive value 5/4. Hence the given original LPP does not possess any feasible solution.

Activity

1. Use two phase method to solve the following LP problems;
   1. *Min Z* = *x*1 − 2*x*2 − 3*x*3

subject to

Table 4.13: Optimal but not Feasible Solution

|  |  |  |  |
| --- | --- | --- | --- |
|  |  | *C*j −→ | 0 0 0 -1 -1 |
| *C*B | B | *b*(= *x*B) | *x*1 *x*2 *x*3 *A*1 *A*1 |
| -1  0 | *A*1  *x*3 | 5/4  1/4 | -7/2 -5/4 0 1 0  1/2 3/4 1 0 1 |
| *Z*∗ = −3 |  | *Z*j = | 7/2 5/4 0 -1 -1 |
|  |  | *C*j − *Z*j | -7/2 -5/4 0 0 0 |

−2*x*1 + *x*2 + 2*x*3 = 2 2*x*1 + 3*x*2 + 2*x*3 = 1

and

*x*1*, x*2*, x*3 ≥ 0

* 1. *Min Z* = 2*x*1 + *x*2 + *x*3

subject to

4*x*1 + 6*x*2 + 3*x*3 = 8

3*x*1 − 6*x*2 − 4*x*3 = 1

2*x*1 + 3*x*2 − 5*x*3 = 4

and

*x*1*, x*2*, x*3 ≥ 0

* + 1. The Big - M Method

The Big - M method is another method of removing artiﬁcial variables from the basis. In this method we assign coeﬃcients to artiﬁcial variables, undesirable from the objective function. If objective function *Z* is to be minimized, then a very large positive price (called penalty) is assigned to each artiﬁcial variable. Similarly, if *Z* is to be maximized, then a very large negative price (also called penalty) is assigned to each of these variables. The penalty will designated by

−*M* for a maximization problem and +*M* for a minimization problem, where

*M >* 0. The following are steps of the Algorithm for solving LPP by the Big - M method;

1. Express the LPP in the standard form by adding slack variables, surplus variables and artiﬁcial variables. Assign a zero coeﬃcient to both slack and surplus variables and a very large positive coeﬃcient +*M* (for min. case) and −*M* (for max. case) to artiﬁcial variable in the objective function.
2. The initial basic feasible solution is obtained by assigning zero value to orig- inal variables.
3. Calculate the value of *Cj* − *Zj* in last row of simplex table and examine these values.
   * If all *Cj* − *Zj* ≥ 0 then the current basic feasible solution is optimal.
   * If for a column *k, Ck* − *Zk* is most negative and all entries in this column are negative, then the problem has unbounded optimal solution.
   * If one or more *Cj* − *Zj <* 0 (minimization case), then select the variable to enter into the basis with the largest negative *Cj* − *Zj* value. That is *Ck* − *Zk* = *Min*{*Cj* − *Zj*} : *Cj* − *Zj <* 0.
4. Determine the key row and key element in the same manner as discussed in the simplex algorithm of the maximization case.

Remarks

At any iteration of the simplex algorithm any one of the following cases may arise;

1. If at least one artiﬁcial variable is present in the basis with zero coeﬃcient of *M* in each case *Cj* −*Zj* ≥ 0, then the given LPP has no solution. That is, the current basic feasible solution is degenerate.
2. If at least one artiﬁcial variable is present in the basis with positive value and the coeﬃcient of *M* in each *Cj* − *Zj* ≥ 0, then given LPP has no optimum basic feasible solution. In this case the given LPP has a pseudo optimum basic feasible solution.

Example 1:Solve the following LPP using penalty (Big - M) method;

*Max Z* = *x*1 + 2*x*2 + 3*x*3 − *x*4

subject to

*x*1 + 2*x*2 + 3*x*3 = 15 2*x*1 + *x*2 + 5*x*3 = 20

*x*1 + 2*x*2 + *x*3 + *x*4 = 10

and

Solution:

*x*1*, x*2*, x*3 ≥ 0

Since all constraints of the given LPP are equation, therefore adding only artiﬁcial variables *A*1 and *A*2 in the constraints. The standard form of the problem becomes;

*Max Z* = *x*1 + 2*x*2 + 3*x*3 − *x*4 − *MA*1 − *MA*2

subject to

and

*x*1 + 2*x*2 + 3*x*3 + *A*1 = 15 2*x*1 + *x*2 + 5*x*3 + *A*2 = 20 *x*1 + 2*x*2 + *x*3 + *x*4 = 10

*x*1*, x*2*, x*3*, A*1*, A*2 ≥ 0

The initial basic feasible solution is given in Table4.14 below;

Table 4.14: Initial Solution

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  |  | *C*j | −→ | 1 | 2 | 3 | -1 | -M | -M |  |
| *C*B | B | *b*(= *x*B) | | *x*1 | *x*2 | *x*3 | *x*4 | *A*1 | *A*2 | Min.Ratio |
| -M | *A*1 | 15 | | 1 | 2 | 3 | 0 | 1 | 0 | 15  = 5  3  20  = 4 →  5  10  = 10  1 |
| -M | *A*2 | 20 | | 2 | 1 | 5 | 0 | 0 | 1 |
| -1 | *x*4 | 10 | | 1 | 2 | 1 | 1 | 0 | 0 |
| *Z* = −35*M* − 10 |  | *Z*j = | | -3M-1 | -3M-2 | -8M-1 | -1 | -M | -M |  |
|  |  | *C*j − *Z*j | | 3M+2 | 3M+4 | 8M+4  ↑ | 0 | 0 | 0 |  |

Since the value of *C*3 − *Z*3 in Table 4.14 has largest positive value the variable *x*3 is chosen to enter into the basis. To get an improved basic feasible solution, apply the following row operations and removing *A*2 from the basis.

*R*2(*old*)

*R*2(*new*) →

5(*key element*)

= (4*,* 2*/*5*,* 1*/*5*,* 1*,* 0*,* 0)

*R*1(*new*) → *R*1(*old*) − (3)*R*2(*new*) = (3*,* −1*/*5*,* 7*/*5*,* 0*,* 0*,* 1)

*R*3(*new*) → *R*3(*old*) − (1)*R*1(*new*) = (6*,* 3*/*5*,* 9*/*5*,* 0*,* 1*,* 0)

The improved solution is shown in Table 4.15

Table 4.15: Improved Solution

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  | *C*j −→ | 1 2 3 -1 -M |  |
| *C*B | B | *b*(= *x*B) | *x*1 *x*2 *x*3 *x*4 *A*1 | Min.Ratio |
| -M 3  -1 | *A*1  *x*3 *x*4 | 3  4  6 | -1/5 7/5 0 0 1  2/5 1/5 1 0 0  3/5 9/5 0 1 0 | 3  = 15*/*7 →  7*/*5  4  = 20  1*/*5  6  = 30*/*9  9*/*5 |
| *Z* = −3*M* +6 |  | *Z*j = | (M/5)-3/5 -(7M/5)-6/5 3 -1 -M |  |
|  |  | *C*j − *Z*j | -(M/5)-2/5 (7M/5)+16/5 0 0 0  ↑ |  |

The solution shown in Table 4.15 is not optimal because *C*2 − *Z*2 is positive. Thus, applying the following row operations for entering variable *x*2 into the basis and removing variable *A*1 from the basis.

*R*1(*old*)

*R*1(*new*) →

7*/*5(*key element*)

= (15*/*7*,* −1*/*7*,* 1*,* 0*,* 0)

*R*2(*new*) → *R*2(*old*) − (1*/*5)*R*1(*new*) = (25*/*7*,* 3*/*7*,* 0*,* 1*,* 0)

*R*3(*new*) → *R*3(*old*) − (9*/*5)*R*1(*new*) = (15*/*7*,* 6*/*5*,* 0*,* 0*,* 1)

The new solution is shown in Table 4.16

Table 4.16: Improved Solution

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  | *C*j −→ | 1 2 3 -1 |  |
| *C*B | B | *b*(= *x*B) | *x*1 *x*2 *x*3 *x*4 | Min.Ratio |
| 2  3  -1 | *x*2 *x*3  *x*4 | 15/7  25/7  15/7 | -1/7 1 0 0  3/7 0 1 0  6/7 0 0 1 | -  25*/*7  = 25*/*3  3*/*7  15*/*7  = 5*/*2 →  6*/*7 |
| *Z* = 90*/*7 |  | *Z*j | 1/7 2 3 -1 |  |
|  |  | *C*j − *Z*j | 6/7 0 0 0  ↑ |  |

Again, the solution shown in Table 4.16 is not optimal. Thus, applying the following row operations by entering *x*1 into the basis and removing variable *x*4 from the basis.

*R*3(*new*) →

*R*3(*old*) 6*/*7(*key element*)

= (15*/*6*,* 1*,* 0*,* 0*,* 7*/*6)

*R*2(*new*) → *R*2(*old*) − (3*/*7)*R*3(*new*) = (15*/*6*,* 0*,* 0*,* 1*,* −1*/*2)

*R*1(*new*) → *R*1(*old*) − (−1*/*7)*R*3(*new*) = (15*/*6*,* 0*,* 1*,* 0*,* 1*/*6)

The new solution is shown in Table 4.17

Table 4.17: Optimal Solution

|  |  |  |  |
| --- | --- | --- | --- |
|  |  | *C*j −→ | 1 2 3 -1 |
| *C*B | B | *b*(= *x*B) | *x*1 *x*2 *x*3 *x*4 |
| 2  3  1 | *x*2  *x*3 *x*1 | 15/6  15/6  15/6 | 0 1 0 1/6  0 0 1 -1/2  1 0 0 7/6 |
| *Z* = 15 |  | *Z*j | 1 2 3 0 |
|  |  | *C*j − *Z*j | 0 0 0 -1 |

Since all *Cj* − *Zj* ≤ 0 in Table 4.17. Thus, an optimal solution has been arrived with values of variables as *x*1 = 15*/*6*, x*2 = 15*/*6*, x*3 = 15*/*6*, x*4 = 0 and *Max Z* = 15.

Example 2:Solve the following LPP using penalty (Big - M) method;

*Main Z* = 600*x*1 + 500*x*2

subject to

2*x*1 + *x*2 ≥ 80

*x*1 + 2*x*2 ≥ 60

and

*x*1*, x*2 ≥ 0

Solution:

By introducing surplus variables *S*1 and *S*2 and artiﬁcial variables *A*1 and *A*2

in the constraints. The standard form of the problem becomes;

*Main Z* = 600*x*1 + 500*x*2 + 0*S*1 + 0*S*2 + *MA*1 + *MA*2

subject to

2*x*1 + *x*2 − *S*1 + *A*1 = 80

*x*1 + 2*x*2 − *S*2 + *A*2 = 60

and

*x*1*, x*2*, S*1*, S*2*, A*1*, A*2 ≥ 0

The initial basic feasible solution is obtained by setting *x*1 = *x*2 = *S*1 = *S*2 = 0 as shown in Table4.18;

Table 4.18: Initial Solution

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  |  | *C*j | −→ | 600 | 500 | 0 | 0 | M | M |  |
| *C*B | B | *b*(= *x*B) | | *x*1 | *x*2 | *S*1 | *S*2 | *A*1 | *A*2 | Min.Ratio |
| M | *A*1 | 80 | | 2 | 1 | -1 | 0 | 1 | 0 | 80  = 80  1  60  = 30 →  2 |
| M | *A*2 | 60 | | 1 | 2 | 0 | -1 | 0 | 1 |
| *Z* = 140*M* |  | *Z*j | | 3M | 3M | -M | -M | M | M |  |
|  |  | *C*j − *Z*j | | 600-3M | 500-3M  ↑ | M | M | 0 | 0 |  |

Since the value of *C*2 − *Z*2 in Table 4.18 has largest negative value, therefore enter variable *x*2 to replace basic variable *A*2 into the basis. To get an improved basic feasible solution, apply the following row operations.

*R*2(*old*)

*R*2(*new*) →

2(*key element*)

= (30*,* 1*/*2*,* 1*,* 0*,* −1*/*2*,* 0)

*R*1(*new*) → *R*1(*old*) − (1)*R*2(*new*) = (50*,* 3*/*2*,* 0*,* −1*,* 1*/*2*,* 1)

The improved solution is shown in Table 4.19

The solution shown in Table 4.19 is not optimal because *C*1 − *Z*1 is largest negative. Thus, applying the following row operations by entering variable *x*1 into the basis and removing variable *A*1 from the basis.

*R*1(*old*)

*R*1(*new*) →

3*/*2(*key element*)

= (100*/*3*,* 1*,* 0*,* −2*/*3*,* 1*/*3)

*R*2(*new*) → *R*2(*old*) − (1*/*2)*R*1(*new*) = (40*/*3*,* 0*,* 1*,* 1*/*3*,* −2*/*3)

Table 4.19: Improved Solution

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  |  | *C*j | −→ | 600 | 500 | 0 | 0 | M |  |
| *C*B | B | *b*(= *x*B) | | *x*1 | *x*2 | *S*1 | *S*2 | *A*1 | Min.Ratio |
| M | *A*1 | 50 | | 3/2 | 0 | -1 | 1/2 | 1 | 50  = 33*.*33 →  3*/*2  30  = 60  1*/*2 |
| 500 | *x*2 | 30 | | 1/2 | 1 | 0 | -1/2 | 0 |
| *Z* = 15000 + 50*M* |  | *Z*j | | (3M/2)+250 | 500 | -M | (M/2)-250 | M |  |
|  |  | *C*j − *Z*j | | 350-3M  ↑ | 0 | M | 250-M/2 | 0 |  |

Table 4.20: Optimal Solution

|  |  |  |  |
| --- | --- | --- | --- |
|  |  | *C*j −→ | 600 500 0 0 |
| *C*B | B | *b*(= *x*B) | *x*1 *x*2 *S*1 *S*2 |
| 600  500 | *x*1  *x*2 | 100/3  40/3 | 1 0 -2/3 1/3  0 1 1/3 -2/3 |
| *Z* = 80*,* 000*/*3 |  | *Z*j | 600 500 -700/3 -400/3 |
|  |  | *C*j − *Z*j | 0 0 700/3 400/3 |

The new solution is shown in Table 4.20

In Table 4.20, all the numbers in the *Cj* − *Zj* row are either zero or positive and also both artiﬁcial variables have been reduced to zero, an optimum solution has been arrived at with *x*1 = 100*/*3*, x*2 = 40*/*3 and total minimum cost, *Z* = 80*,* 000*/*3.

Activity

1. Use Big - M method to solve the following LPP.

*Max Z* = 8*x*1 + 15*x*2 + 25*x*3 + *x*4

subject to

*x*1 + 2*x*2 + 3*x*3 = 15 2*x*1 + *x*2 + 5*x*3 = 20

*x*1 + 2*x*2 + *x*3 + *x*4 = 10

and

*x*1*, x*2*, x*3*, x*4 ≥ 0

1. Use Big - M method to solve the following LPP.

*Max Z* = 12*x*1 + 20*x*2 + 18*x*3 + 40*x*4

subject to

4*x*1 + 9*x*2 + 7*x*3 + 10*x*4 ≤ 6000

*x*1 + *x*2 + 3*x*3 + 40*x*4 ≤ 4000

and

*x*1*, x*2*, x*3*, x*4 ≥ 0

* 1. Degeneracy in Simplex Method

A basic feasible solution of a simplex method is said to be degenerate basic feasible solution if at least one of the basic variable is zero and at any iteration of the simplex method more than one variable is eligible to leave the basis and hence the next simplex iteration produces a degenerate solution in which at least one basic variable is zero. This concept is known as tie.

A situation may arise at any iteration when two or more columns may have exactly the same *Cj* − *Zj* value (+ve or -ve depending on the type of LPP). In order to break this tie, the selection for key column (entering variable) can be made arbitrary,. However, the number of iterations required to arrive at the optimal solution can be minimized by adopting the following rules;

If there is a tie between two decision variables, then the selection can be made arbitrarily.

If there is a tie between decision variable and slack (or surplus) variable, then select the decision variable to enter into the basis ﬁrst.

If there is a tie between two slack (or surplus) variables, then selection can be made arbitrarily.

Again, while solving LPP the situation may arise in which there is a tie between two or more basic variables for leaving the basis i.e minimum ratio to identify the basic variable to leave the basis is not unique or values of one or more basic

variables in the solution values column (*xB*) become equal to zero. This causes the problem of degeneracy. However, if minimum ration is zero, then the iterations of simplex method are repeated (cycle) indeﬁnitely without arriving at the optimal solution.

In most of the cases when there is a tie in the minimum ratios, the selection is made arbitrarily. However, the number of iterations required to arrive at the optimal solution can be minimized by applying the following rules;

Divide the coeﬃcients of slack variables in the simplex table where degener- acy is detected by the corresponding positive numbers of the key column in the row, starting from left to right.

The row which contains smallest ratio comparing from left to right column- wise becomes the key row.

Remark:When there is a tie between a slack and artiﬁcial variables to leave the basis, the preference shall be given to artiﬁcial variable to leave the basis and there is no need to apply the procedure for resolving degeneracy under such cases.

Example: Solve the following LPP

*Max Z* = 3*x*1 + 9*x*2

subject to the constraints

and

Solution:

*x*1 + 4*x*2 ≤ 8

*x*1 + 2*x*2 ≤ 4

*x*1*, x*2 ≥ 0

Adding slack variables *S*1 and *S*2 to the constraints, the problem can be expressed as;

*Max Z* = 3*x*1 + 9*x*2 + 0*S*1 + 0*S*2

subject to the constraints

and

*x*1 + 4*x*2 + *S*1 = 8

*x*1 + 2*x*2 + *S*2 = 4

*x*1*, x*2*, S*1*, S*2 ≥ 0

The initial basic feasible solution is given in Table 4.21

Table 4.21: Initial Solution

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  |  | *C*j | −→ | 3 | 9 | 0 | 0 |  | |
| *C*B | B | *b*(= *x*B) | | *x*1 | *x*2 | *S*1 | *S*2 | Min.Ratio | |
| 0 | *S*1 | 8 | | 1 | 4 | 1 | 0 | 8 | = 2  = 2 |
| 4 |
| 0 | *S*2 | 4 | | 1 | 2 | 0 | 1 | 4 |
| 2 |
| *Z* = 0 |  | *Z*j | | 0 | 0 | 0 | 0 |  | |
|  |  | *C*j − *Z*j | | 3 | 9  ↑ | 0 | 0 |  | |

from the Table 4.21, *C*2 − *Z*2 is the largest positive value, therefore variable *x*2 is selected to enter into the basis. However, both variables *S*1 and *S*2. This is an indication of the existence of degeneracy. To obtain the unique key row, apply the following procedure for resolving degeneracy.

Write coeﬃcients of the slack variables as follows;

|  |  |  |
| --- | --- | --- |
| Row | Column  *S*1 | *S*2 |
| *S*1 | 1 | 0 |
| *S*2 | 0 | 1 |

Dividing the coeﬃcients by the corresponding element of the key column, we obtain the following ratios;

|  |  |  |
| --- | --- | --- |
| Row | Column  *S*1 | *S*2 |
| *S*1 | 1/4=1/4 | 0/4=0 |
| *S*2 | 0/2=0 | 1/2=1/2 |

Comparing the ratios of the previous step from left to right column-wise, the minimum ratio occurs for the second row. Therefore, the variable *S*2 is selected to leave the basis. The new solution is obtain by performing the following row operations and shown in Table 4.22

*R*2(*old*)

*R*2(*new*) −→

2(*keyelement*)

= (2*,* 1*/*2*,* 1*,* 0*,* 1*/*2)

*R*1(*new*) −→ *R*1(*old*) − 4*R*2(*new*) = (0*,* −1*,* 0*,* 1*,* −2)

Table 4.22: Optimal Solution

|  |  |  |  |
| --- | --- | --- | --- |
|  |  | *C*j −→ | 3 9 0 0 |
| *C*B | B | *b*(= *x*B) | *x*1 *x*2 *S*1 *S*2 |
| 0  9 | *S*1  *x*2 | 0  2 | -1 0 1 -2  1/2 1 0 1/2 |
| *Z* = 18 |  | *Z*j | 9/2 9 0 9/2 |
|  |  | *C*j − *Z*j | -3/2 0 0 -9/2 |

Since all *Cj* − *Zj* ≤ 0 in Table 4.22. Therefore, an optimal solution is arrived at *x*1 = 0*, x*2 = 2 and Max *Z* = 18.

* 1. Types of Linear Programming Solution
     1. Alternative (Multiple) Optimal Solution

The alternative optimal solution can be obtained by considering the *Cj* −*Zj* row of the simplex table. We know that an optimal solution to a maximization problem is reached if all *Cj* − *Zj* ≤ 0. What will happen if *Cj* − *Zj* = 0 for some non-basic variable columns in the optimal simplex table? Each entry in the *Cj* −*Zj* indicates the contribution per unit of a particular variable in the objective function value if is entered into the basis. Thus, if a non-basic variable corresponding to which *Cj* − *Zj* = 0 is entered into the basis, a new solution will be arrived at but the value of the objective function will not change.

Example: Solve the following LPP;

*Max Z* = 6*x*1 + 4*x*2

subject to the constraints

2*x*1 + 3*x*2 ≤ 30

3*x*1 + 2*x*2 ≤ 24

*x*1 + *x*2 ≥ 3

and

*x*1*, x*2 ≥ 0

Solution:

Adding slack variables *S*1*, S*2, surplus variable *S*3 and artiﬁcial variable *A*1

in the constraint set the LPP becomes;

*Max Z* = 6*x*1 + 4*x*2 + 0*S*1 + 0*S*2 + *S*3 − *MA*1

subject to the constraints

2*x*1 + 3*x*2 + *S*1 = 30

3*x*1 + 2*x*2 + *S*2 = 24

*x*1 + *x*2 − *S*3 + *A*1 = 3

and

*x*1*, x*2*, S*1*, S*2*, S*3*, A*1 ≥ 0

The optimal solution for this LPP is presented in Table 4.23

Table 4.23: Optimal Solution

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  | *C*j −→ | 6 4 0 0 0 |  |
| *C*B | B | *b*(= *x*B) | *x*1 *x*2 *S*1 *S*2 *S*3 | Min.Ratio |
| 0  0  6 | *S*1 *S*3  *x*1 | 14  5  8 | 0 5/3 1 -2/3 0  0 -1/3 0 1/3 1  1 2/3 0 1/3 0 | 14  = 42*/*5 →  15*/*3  -  8  = 12  2*/*3 |
| *Z* = 48 |  | *Z*j | 6 4 0 2 0 |  |
|  |  | *C*j − *Z*j | 0 0 0 -2 0  ↑ |  |

The optimal solution shown in Table 4.23 is *x*1 = 8*, x*2 = 0 and Max Z=48.

From the Table 4.23, *C*2 − *Z*2 = 0 corresponding to a non-basic variable, *x*2 = 0. Thus, an alternative optimal solution can also be obtained by entering variable *x*2 into the basis and removing *S*1 from the basis. The new solution is shown in Table 4.24

The optimal solution shown in Table 4.24 is *x*1 = 12*/*5*, x*2 = 42*/*5 and Max Z=48.

Further observe that in Table 4.24, *C*3 − *Z*3 = 0 and variable *S*1 is not in the basis. This again indicates that an alternative optimal solution exists, thus for each alternative solution (inﬁnite number of solutions) the value of objective function will remain the same.

Table 4.24: Alternative Solution

|  |  |  |  |
| --- | --- | --- | --- |
|  |  | *C*j −→ | 6 4 0 0 0 |
| *C*B | B | *b*(= *x*B) | *x*1 *x*2 *S*1 *S*2 *S*3 |
| 4  0  6 | *x*2 *S*3  *x*1 | 42/5  39/5  12/5 | 0 1 3/5 -2/5 0  0 0 1/5 1/5 1  1 0 -2/5 3/5 0 |
| *Z* = 48 |  | *Z*j | 6 4 0 2 0 |
|  |  | *C*j − *Z*j | 0 0 0 -2 0 |

* + 1. Unbounded Solution

In maximization LPP, if *Cj* − *Zj >* 0(*Cj* − *Zj <* 0 for a maximization case) for a column not in the basis and all entries in this column are negative, then for determining key row, we have to calculate minimum ratio corresponding to each basic variable having negative or zero value in the denominator. Negative value in the denominator can not be considered, as it would indicate the entry of non-basic variable in the basis with a negative value (an infeasible solution will occur). A zero value in the denominator would result in ratio having a +∞. This implies that the entering variable could be increased indeﬁnitely with any of the current basic variables being removed from the basis. In general, an unbounded solution occurs due to wrong formulation of the problem within the constraint set, and thus needs reformulation.

Example: Solve the following LPP;

*Max Z* = 3*x*1 + 5*x*2

subject to the constraints

and

Solution:

*x*1 − 2*x*2 ≤ 6

*x*1 ≤ 10

*x*2 ≥ 1

*x*1*, x*2 ≥ 0

Adding slack variables *S*1*, S*2, surplus variable *S*3 and artiﬁcial variable *A*1

in the constraint set the LPP becomes;

*Max Z* = 3*x*1 + 5*x*2 + 0*S*1 + 0*S*2 + 0*S*3 − *MA*1

subject to the constraints

*x*1 − 2*x*2 + *S*1 = 6 *x*1 + *S*2 = 10 *x*2 − *S*3 + *A*11

and

*x*1*, x*2*, S*1*, S*2*, S*3*, A*1 ≥ 0

The initial solution to this LPP is shown in Table 4.25

Table 4.25: Initiall Solution

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  | *C*j −→ | 3 5 0 0 0 -M |  |
| *C*B | B | *b*(= *x*B) | *x*1 *x*2 *S*1 *S*2 *S*3 *A*1 | Min.Ratio |
| 0  0  -M | *S*1 *S*2  *A*1 | 6  10  1 | 1 -2 1 0 0 0  1 0 0 1 0 0  0 1 0 0 -1 1 | -  -  1  = 1 →  1 |
| *Z* = −*M* |  | *Z*j | 0 -M 0 0 M -M |  |
|  |  | *C*j − *Z*j | 3 5+M 0 0 -M 0  ↑ |  |

From Table 4.25, *C*2 − *Z*2 has largest positive value, thus variable *x*2 enters the basis and *A*1 leaves the basis. The new solution is shown in Table 4.26

Table 4.26: Improved Solution

|  |  |  |  |
| --- | --- | --- | --- |
|  |  | *C*j −→ | 3 5 0 0 0 -M |
| *C*B | B | *b*(= *x*B) | *x*1 *x*2 *S*1 *S*2 *S*3 *A*1 |
| 0  0  5 | *S*1 *S*2  *x*2 | 8  10  1 | 1 0 1 0 -2 2  1 0 0 1 0 0  0 1 0 0 -1 1 |
| *Z* = 5 |  | *Z*j | 0 5 0 0 -5 5 |
|  |  | *C*j − *Z*j | 3 0 0 0 5 -M-5 |

From the Table 4.26, *C*1 − *Z*1 = 3 and *C*5 − *Z*5 = 5 entries are positive and *C*5 − *Z*5 ≥ *C*1 − *Z*1. Therefore, variable *S*3 should enter into the basis. Here it may be noted that coeﬃcients in the ’*S*3’ column are all negative or zero. This

indicates that *S*3 cannot be entered into the basis. However, the value of *S*3 can be increased inﬁnitely without removing any one of the basic variables. Further, since *S*3 is associated with *x*1 in the third constraint, *x*1 will also be increased inﬁnitely because it can be expressed as *x*1 = 1 + *S*3 − *A*1. Hence, the solution to the given LPP is unbounded.

* + 1. Infeasible Solution

In the ﬁnal simplex table, if atleast one of the artiﬁcial variable appears with a positive value, no feasible solution exists, because it is not possible to remove such an artiﬁcial variable from the basis using the simplex algorithm. When an infeasible solution exists, the LP Model should be reformulated. This may be because of the fact that the model is either improperly formulated or two or more of the constraints are incompatible.

Example:

*Max Z* = 6*x*1 + 4*x*2

subject to the constraints

*x*1 + *x*2 ≤ 5

*x*2 ≥ 8

and

*x*1*, x*2 ≥ 0

Solution:

By adding slack, surplus and artiﬁcial variables, the LPP becomes;

*Max Z* = 6*x*1 + 4*x*2 + 0*S*1 + 0*S*2 − *MA*1

subject to the constraints

*x*1 + *x*2 + *S*1 = 5

*x*2 − *S*2 + *A*1 = 8

and

*x*1*, x*2*, S*1*, S*2*, A*1 ≥ 0

The initial solution to this LPP is shown in Table 4.27

Table 4.27: Initial Solution

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  | *C*j −→ | 6 4 0 0 -M |  |
| *C*B | B | *b*(= *x*B) | *x*1 *x*2 *S*1 *S*2 *A*1 | Min.Ratio |
| 0  -M | *S*1  *A*1 | 5  8 | 1 1 1 0 0  0 1 0 -1 1 | 5 = 5 →  1  8 = 8  1 |
| *Z* = −8*M* |  | *Z*j | 0 -M 0 M -M |  |
|  |  | *C*j − *Z*j | 6 4+M 0 -M 0  ↑ |  |

Table 4.28:

|  |  |  |  |
| --- | --- | --- | --- |
|  |  | *C*j −→ | 6 4 0 0 -M |
| *C*B | B | *b*(= *x*B) | *x*1 *x*2 *S*1 *S*2 *A*1 |
| 4  -M | *x*2  *A*1 | 5  3 | 1 1 1 0 0  -1 0 -1 -1 1 |
| *Z* = 20 − 3*M* |  | *Z*j | 4+M 4 4+M M -M |
|  |  | *C*j − *Z*j | 2-M 0 -4-M -M 0 |

Variable *x*2 enters the basis and *S*1 leaves the basis. The new solution is shown in Table 4.28

Since all *Cj* − *Zj* ≤ 0, the solution shown in Table 4.28 ia optimal. But this solution is not feasible for the given problem since it has *x*1 = 0 and *x*2 = 5 (recall that in the second constraint *x*2 ≥ 8). The fact that artiﬁcial variable *A*1 = 3 is in the solution also indicates that the ﬁnal solution violates the second constraint.

DUALITY IN LINEAR PROGRAMMING

* 1. Introduction

The term dual in general sense implies two or double. In the context of linear programming duality implies that each LPP can be analysed in two diﬀerent ways but having equivalent solution. Moreover, whenever the LPP contains a large number of constraints and a smaller number of variables then the labour of computational can be considerably reduced by converting it into the dual and then solve it. Every LPP is associated with another LPP called the dual based on the same data. The original problem is called the primal.

* 1. Formulation of Dual Linear Programming Problem

Let the primal LPP be;

*Max Zx* = *c*1*x*1 + *c*2*x*2 + *...* + *cnxn*

subject to the constraints

*a*11*x*1 +*a*12*x*2 + *...* + *a*1*nxn* ≤ *b*1 *a*21*x*1 +*a*22*x*2 + *...* + *a*2*nxn* ≤ *b*2

.**. .**. **.**.

*am*1*x*1 +*am*2*x*2 + *...* + *amnxn* ≤ *bm*

and

*x*1*, x*2*,..., xn* ≥ 0

Then the corresponding dual is deﬁned as:

*Min Zy* = *b*1*y*1 + *b*2*y*2 + *...* + *bmym*

subject to the constraints

*a*11*y*1 +*a*12*y*2 + *...* + *a*1*nym* ≤ *c*1 *a*21*y*1 +*a*22*y*2 + *...* + *a*2*nym* ≤ *c*2

.**. .**. **.**.

*an*1*y*1 +*an*2*y*2 + *...* + *amnym* ≤ *cn*

*y*1*, y*2*,..., ym* ≥ 0

* + 1. Rules for Constructing the Duality from Primal
       1. Change the objective of maximization in the primal into minimization in the dual and vice-versa.
       2. For a maximization primal with all ≤ type constraints, there exists a min- imization dual problem with all ≥ type constraints and vice-versa. The inequality sign for non-negativity constraint is unreversed.
       3. The number of variables in the primal will be the number of constraints in the dual and vice-versa.
       4. The cost of coeﬃcients *c*1*, c*2*,..., cn* in the objective function of the primal will be the RHS constant of the constraints in the dual and vice-versa.
       5. For the constraints of dual, transpose the body matrix of the primal problem.
       6. If the *ith* primal variable is unrestricted in sign, then the *jth* dual constraint is = type and vice-versa.

Example 1: Write the dual of the following LPP;

*Max Zx* = 3*x*1 + *x*2 + *x*3

Subject to

and

4*x*1 − *x*2 ≤ 8

8*x*1 + *x*2 + 3*x*3 ≥ 12

5*x*1 − 6*x*3 ≤ 13

*x*1*, x*2*, x*3 ≥ 0

Solution: Let *y*1*, y*2 and *y*3 be the dual variables, then the corresponding dual is;

*Min Zy* = 8*y*1 + 12*y*2 + 13*y*3

Subject to

4*y*1 − 8*y*2 + 5*y*3 ≥ 13

−*y*1 − *y*2 ≥ −1

−3*y*1 − 6*y*3 ≥ 1

*y*1*, y*2*, y*3 ≥ 0

Example 2: Write the dual of the following LPP;

*Min Zx* = 3*x*1 − 2*x*2 + 4*x*3

Subject to

and

3*x*1 + 5*x*2 + 4*x*3 ≥ 7

6*x*1 + *x*2 + 3*x*3 ≥ 4

7*x*1 − 2*x*2 − *x*3 ≤ 10 *x*1 − 2*x*2 + 5*x*3 ≥ 3 4*x*1 + 7*x*2 − 2*x*3 ≥ 2

*x*1*, x*2*, x*3 ≥ 0

Solution:Since the objective function is of minimization type all inequalities have to be changed to ≥ type. Constraint No:3 will change to;

−7*x*1 + 2*x*2 + *x*3 ≥ −10

Let *y*1*, y*2*, y*3*, y*4 and *y*5 are dual variable corresponding to ﬁve primal constraints,thus the dual to this LPP is;

*Max Zy* = 7*y*1 + 4*y*2 − 10*y*3 + 3*y*4 + 2*y*5

Subject to

and

3*y*1 + 6*y*2 − 7*y*3 + *y*4 + 4*y*5 ≤ 3

5*y*1 + *y*2 + 2*y*3 − 2*y*4 + 7*y*5 ≤ −2

4*y*1 + 3*y*2 + *y*3 + 5*y*4 − 2*y*5 ≤ 4

*y*1*, y*2*, y*3*, y*4*, y*5 ≥ 0

Example 3:Obtain the dual of the following LPP;

*Max Zx* = *x*1 − 2*x*2 + 3*x*3

Subject to

−2*x*1 + *x*2 + 3*x*3 = 2 2*x*1 + 3*x*2 + 4*x*3 = 1

*x*1*, x*2*, x*3 ≥ 0

Solution: Since both the primal constraints are = type, the corresponding dual variables *y*1*, y*2 will be unrestricted in sign, the dual to this LPP is;

*Min Zy* = 2*y*1 + *y*2

Subject to

−2*y*1 + 2*y*2 ≥ 1 *y*1 + 3*y*2 ≥ −2 3*y*1 + 4*y*2 ≥ 3

and

*y*1*, y*2 *unrestricted in sign*

* + 1. Primal - Dual Relationship

A summary of the general relationship between primal and dual LPP is given in Table 5.1

Table 5.1: Primal-Dual Relationship

|  |  |
| --- | --- |
| If Primal | Then Dual |
| i)Objective is to maximise | i)Objective is to minimise |
| ii)Variable *x*j | ii)Constraint j |
| iii)Constraint i | iii)Variable *y*i |
| iv)Variables *x*j unrestricted insign | iv)Constraint j is = type |
| v)Constraint i is = type | v)Variable *y*i is unrestricted in sign |
| vi)≤ type constraints | vi)≥ type constraints |
| vii)*x*j unrestricted in sign | vii)*j*th constraint is an equatio |

* 1. Standard Results on Duality

You can make a proof of the following standard results;

1. The dual of the dual LPP is again the primal problem.
2. If either the primal or the dual has an unbounded objective function value, the other problem has no feasible solution.
3. If either the primal or dual problem has a ﬁnite optimal solution, the other one also possesses the same, and the optimal value of the objective function of the two problems are equal i.e. Max *Zx* = Min *Zy*. This analytical result is known as the fundamental primal-dual relationship.
4. Complementary slackness property of primal-dual relationship states that for a positive basic variable in the primal, the corresponding dual variable will be equal to zero. Alternatively, for a non-basic variable in the primal (which is zero), the corresponding dual variable will be basic and positive.
   1. Signiﬁcant of Duality

The importance of dual LPP is in terms of the information which it provides about the value of the resources. The economic analysis is concerned in deciding whether or not to secure more resources and how much to pay for these additional resources. The signiﬁcance of the study of dual is as follows;

1. The dual variables provide the decision-maker a basis for deciding how much to pay for additional units of resources.
2. The maximum amount that should be paid for one additional unit of a re- source is called its shadow price (also called simplex multiplier).
3. The total marginal value of the resources equals the optimal objective func- tion value. The dual variables equal the marginal value of resources (shadow prices).
   1. Advantages of Duality
4. It is advantageous to solve the dual of primal having less number of con- straints, because the number of constraints usually equals the number of iterations required to solve the problem.
5. It avoids the necessity for adding surplus or artiﬁcial variables and solves the problem quickly (the technique is known as the primal-dual method). In economics, duality is useful in the formulation of the input and output systems. It is also in physics, engineering, mathematics, etc.
6. The dual variables provide an important economic interpretation of the ﬁnal solution of an LPP.
7. It is quite useful when investigating changes in the parameters of an LPP (the technique is known as the sensitivity analysis).
8. Duality is used to solve an LPP by the simplex method in which the initial solution is infeasible (the technique is known as the dual simplex method)

Problem:

Write the dual of the following primal LPP;

1.

Subject to

and

*Max Zx* = 2*x*1 + 5*x*2 + 6*x*3 5*x*1 + 6*x*2 − *x*3 ≤ 3

−2*x*1 + *x*2 + 4*x*3 ≤ 4

*x*1 − 5*x*2 + 3*x*3 ≤ 1

−3*x*1 − 3*x*2 + 7*x*3 ≤ 6

*x*1*, x*2*, x*3 ≥ 0

2.

Subject to

and

*Max Zx* = 2*x*1 + 3*x*2 + *x*3

4*x*1 + 3*x*2 + *x*3 = 6

*x*1 + 2*x*2 + 5*x*3 = 4

*x*1*, x*2*, x*3 ≥ 0

3.

Subject to

and

*Min Zx* = 2*x*1 + 3*x*2 + *x*3

2*x*1 + 3*x*254*x*3 ≥ 2

3*x*1 + *x*2 + 5*x*3 = 3

*x*1 + 4*x*2 + 6*x*3 ≤ 5

*x*1*, x*2 ≥ 0*, x*3 *is unrestricted.*