#  Transportation problem

In the application of linear programming techniques, the transportation problem was probably one of the ﬁrst signiﬁcant problems studied. The problem can be expressed by the formulation of a linear model, and it can be solved using the simplex algorithm. However, and because of the special structure of the linear model, it can be solved with a more efﬁcient method which is the topic of this chapter.

The transportation problem deals with the transportation of any product from *m*origins,*O* 1*, . . . , Om*, to*n*destinations,*D* 1*, . . . , Dn*, with the aim of minimizing the total distribution cost, where:

* + - The origin*O i* has a supply of*a i* units,*i*= 1*, . . . , m*.
		- The destination*D j* has a demand for*b j* units to be delivered from the ori- gins,*j*= 1*, . . . , n*.
		- *c ij* is the cost per unit distributed from the origin*O i* to the destination*D j*,

*i*= 1*, . . . , m, j*= 1*, . .*

In mathematical terms, the above problem can be expressed as ﬁnding a set of *xij*’s,*i*= 1*, . . . , m, j*= 1*, . . . , n*, to meet supply and demand requirements at a minimum distribution cost. The corresponding linear model is:

*m n*

min*z*=

� � *cij xij*

subject to

*i*=1 *j*=1

� *xij* ≤*a i, i*= 1*, . . . , m*

*j*=1 *m*

� *xij* ≥*b j, j*= 1*, . . . , n*

*i*=1

*xij* ≥0*, i*= 1*, . . . , m, j*= 1*, . . . , n*

Thus, the problem is to determine*x ij*, the number of units to be transported from*O i* to*D j*, so that supplies will be consumed and demands satisﬁed at an overall minimum cost.

The ﬁrst*m*constraints correspond to the supply limits, and they express that the supply of commodity units available at each origin must not be exceeded. The next*n*constraints ensure that the commodity unit requirements at destinations will be satisﬁed. The decision variables are deﬁned positive, since they represent the number of commodity units transported.

The transportation problem in standard form is shown below:

*m n*

min*z*=

� � *cij xij*

subject to

*i*=1 *j*=1

*n*

� *xij* =*a i, i*= 1*, . . . , m*

*j*=1 *m*

� *xij* =*b j, j*= 1*, . . . , n*

*i*=1

*xij* ≥0*, i*= 1*, . . . , m, j*= 1*, . . . , n*

**Example.**Two bread factories,*O* 1 and*O* 2, make the daily bread in a city. The bread is delivered to the three bakeries of the city:*D* 1*, D*2 and*D* 3. The supplies

* 1. The transportation problem 153

of bread factories, the demands of bakeries and the per unit transportation costs are displayed in the following graph:

1000

9

2500 *O*2

4

10

2000

6

10

2000

8

1500

*D*3

*D*2

*O*1

*D*1

# Matrix format for the Transportation Problem

The relevant data for any transportation problem can be summarized in a matrix format using a tableau called*the transportation costs tableau*(see Figure 5.1). The tableau displays the origins with their supply, the destinations with their de- mand and the transportation per-unit costs.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | *D*1 | *D*2 | · · · | *Dn* | Supply |
| *O*1 | *c*11 | *c*12 | · · · | *c*1*n* | *a*1 |
| *O*2 | *c*21 | *c*22 | · · · | *c*2*n* | *a*2 |
| . | . | . | . . . | . | . |
| *Om* | *cm*1 | *cm*2 | · · · | *cmn* | *am* |
| Demand | *b*1 | *b*2 | · · · | *bn* |  |
|  |  |  |  |  |  |

**Types of TP**

* *Balanced TP*
* *Unbalanced TP*

*A transportation problem is said to be balanced if*

 *Total Demand = Total Supply*

*A transportation problem is said to be Unbalanced if*

 *Total Demand =! Total Supply*

*For ex. (ROW)*

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | 1 | 2 | 3 | Supply |
| 1 | 2 | 4 | 3 | 10 |
| 2 | 6 | 1 | 4 | 20 |
| Demand | 20 | 20 | 20 |  |

* + - Total supply =*a* 1 +*a* 2 = 10 + 20 = 30.
		- Total demand =*b* 1 +*b* 2 +*b* 3 = 20 + 20 + 20 = 60.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | 1 | 2 | 3 | Supply |
| 1 | 2 | 4 | 3 | 10 |
| 2 | 6 | 1 | 4 | 20 |
| 3 | 0 | 0 | 0 | 30 |
| Demand | 20 | 20 | 20 |  |

The demand exceeds the supply. We add the dummy origin3to balance the problem, being*a* 3 = 60−30 = 30its supply. We consider unit transporta- tion costs*c* 31,*c* 32 and*c* 33 to be zero. This leads to the following balanced transportation problem in matrix format:

For Ex (Col)

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | 1 | 2 | 3 | Supply |
| 1 | 3 | 2 | 1 | 50 |
| 2 | 6 | 4 | 4 | 50 |
| Demand | 20 | 20 | 20 |  |

* + Total supply =*a* 1 +*a* 2 = 50 + 50 = 100.
	+ Total demand =*b* 1 +*b* 2 +*b* 3 = 20 + 20 + 20 = 60.

The total supply is higher than the total demand. We add a dummy destination 4, with a demand*b* 4 = 100−60 = 40. The unit transportation costs*c* 14 and*c* 24 are considered as zero. The balanced transportation problem in matrix format is:

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | 1 | 2 | 3 | 4 | Supply |
| 1 | 3 | 2 | 1 | 0 | 50 |
| 2 | 6 | 4 | 4 | 0 | 50 |
| Demand | 20 | 20 | 20 | 40 |  |

## The Northwest Corner method

**Example.**Consider the balanced transportation costs tableau of the example on

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | *D*1 | *D*2 | *D*3 | Supply |
| *O*1 | 8 | 6 | 10 | 2000 |
| *O*2 | 10 | 4 | 9 | 2500 |
| Demand | 1500 | 2000 | 1000 |  |

Supply =2000 + 2500 = 1500 + 2000 + 1000 =Demand

### First iteration.

* + - * **Step 1.**We select the upper left-hand corner cell(1*,*1), which corresponds to the 1st row and the 1st column of the solution tableau (see the symbol∗ entered in the tableau shown below).

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | *D*1 | *D*2 | *D*3 | Supply |
| *O*1 | \* |  |  | 2000 |
| *O*2 |  |  |  | 2500 |
| Demand | 1500 | 2000 | 1000 |  |

* + - * **Step 2.**Assign to the variable*x* 11 the maximum feasible amount consistent with the row and the column requirements, and adjust the supply and the demand requirements.

*x*11 = min{*a* 1*, b*1}= min{2000*,*1500}= 1500

The adjusted supply of the origin*O* 1:*a* 1 = 2000−*x* 11 = 500. The adjusted demand of the destination*D* 1:*b* 1 = 1500−*x* 11 = 0.

The demand of the destination*D* 1 has been satisﬁed; the column1is elimi- nated from further consideration.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | *D*1 | *D*2 | *D*3 | Supply |
| *O*1 | 1500 | \* |  | 2000500 |
| *O*2 |  |  |  | 2500 |
| Demand | 1500 | 2000 | 1000 |  |

* + - * **Step 3.**More than one row or one column remain under consideration in the solution tableau. Therefore, we go to Step 1, and proceed with a new iteration of the algorithm.

**Second iteration.**As in the ﬁrst iteration, in the rows and columns un- der consideration we select the cell in the upper left-hand corner. This time, we select the cell(1*,*2), which corresponds to the 1st row and the 2nd column of the solution tableau (see the symbol∗entered in the tableau shown above).

*x*12 = min{500*,*2000}= 500. We adjust the supply and the demand require- ments:*a* 1 = 500−*x* 12 = 0and*b* 2 = 2000−500 = 1500. The supply of the

origin*O* 1 becomes zero, and the row1is eliminated from further consideration.

The adjusted solution tableau is the following:

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | *D*1 | *D*2 | *D*3 | Supply |
| *O*1 | 1500 | 500 |  | 2000500 |
| *O*2 |  |  |  | 2500 |
| Demand | 1500 | 20001500 | 1000 |  |

Only one row remains under consideration: the second, which corresponds to the origin*O* 2. Thus, the two remaining cells(2*,*2)and(2*,*3)are selected and

the remaining supplies*x* 22 = 1500and*x* 23 = 1000are assigned to the variables associated with them. The algorithm stops; the tableau shows an initial solution to the problem.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | *D*1 | *D*2 | *D*3 | Supply |
|  |  |  |  |  |
| *O*1 | 1500 | 500 |  | 2000 |
| *O*2 |  | 1500 | 1000 | 2500 |
| Demand | 1500 | 2000 | 1000 |  |

The initial solution obtained is basic and feasible.*m*+*n*−1 = 2 + 3−1 = 4

variables have been assigned a positive value; they are the basic variables.

* The solution.

*x*11 = 1500*, x*12 = 500*, x*13 = 0*, x*21 = 0*, x*22 = 1500*, x*23 = 1000*.*

* The total transportation cost.

*z*= (1500×8) + (500×6) + (1500×4) + (1000×9) = 30000*.*�

The Northwest Corner method is a very simple procedure proposed to ﬁnd a basic feasible solution for a transportation problem. However, the unit transporta- tion costs play no role in this method, which simply selects the upper left-hand corner variable and assigns a value to it. We next present another method, which takes the unit transportation costs into account, and which usually results in a basic feasible solution close to the optimal solution.

## Vogel’s approximation method

The main difference between the Northwest Corner method and Vogel’s approx- imation method lays in the criteria used in Step 1 to select a cell in the solution tableau. Instead of selecting the upper left-hand corner, Vogel’s approximation method computes row differences and column differences to select a cell. The row difference and the column difference are deﬁned as follows:

* *RD i* = the arithmetic difference between the smallest and the next smallest unit cost*c ij* which remain under consideration in the row*i*,*i*= 1*, . . . , m*.
* *CD j* = the arithmetic difference between the smallest and the next small- est unit cost*c ij* which remain under consideration in the column*j*,*j*=

1*, . . . , n*.

The row differences and the column differences are used to make a more con- venient selection of a cell in the solution tableau; a selection based on the unit transportation costs. Given a balanced transportation problem, and starting at a solution tableau with all the cells(*i, j*)empty, Vogel’s approximation method con- sists of the following steps, and leads to a reasonably good initial basic feasible solution:

* + - * **Step 1**. Compute for each row*i*and each column*j*the differences*RD i* and *CDj*. Among the rows and columns still under consideration, ﬁnd the one with the largest difference, and ﬁnd in it the cell(*i, j*)with the smallest unit transportation cost*c ij*.
				+ **Step 2**. Assign to the variable*x ij* the maximum feasible amount consistent with the row and the column requirements of that cell, that is, the value:

*xij* = min{*ai, bj*}*.*

At least one of the requirements, the supply or the demand, will then be met. Adjust the supply*a i* and the demand*b j* as follows:

If*a i* happens to be the minimum, then the supply of the origin*O i* becomes zero, and the row*i*is eliminated from further consideration. The demand*b j* is replaced by*b j* −*a i*.

If*b j* happens to be the minimum, then the demand of the destination *Dj* becomes zero, and the column*j*is eliminated from further consid- eration. The supply*a i* is replaced by*a i* −*b j*.

If*a i* =*b j*, then the adjusted values for the supply*a i* and the demand *bj* become both zero. The row*i*and the column*j*are eliminated from further consideration.

* + - * + **Step 3**. Two cases may arise:

If only one row or only one column remains under consideration, then any remaining cells(*i, j*), that is, variables*x ij* associated with these cells, are selected and the remaining supplies are assigned to them. Stop.

Otherwise, go to Step 1.

**Example.**Consider the balanced transportation costs tableau of the example on page 155. Let us apply Vogel’s approximation method to ﬁnd an initial basic feasible solution.

### First iteration.

* + - * + **Step 1**. We compute the row and column differences,*RD i* and*CD j*, in the transportation costs tableau, and ﬁnd the row or the column with the largest difference:max{2*,*5*,*2*,*2*,*1}= 5, so that the second row is chosen. The smallest unit transportation cost in this row ismin{10*,*4*,*9}= 4. Thus, cell (2*,*2)has been selected.

Transportation costs tableau Transportation solution tableau

*RDi*

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | *D*1 | *D*2 | *D*3 | Supply |
| *O*1 | 8 | 6 | 10 | 2000 |
| *O*2 | 10 | 4 | 9 | 2500 |
| Deman. | 1500 | 2000 | 1000 |  |

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | *D*1 | *D*2 | *D*3 | Supply |
| *O*1 |  |  |  | 2000 |
| *O*2 |  | \* |  | 2500 |
| Deman. | 1500 | 2000 | 1000 |  |

2

5

*CDj* 2 2 1

* + - * **Step 2**. Assign to the variable*x* 22 the maximum feasible amount consistent with the row and the column requirements of cell(2*,*2):

*x*22 = min{2500*,*2000}= 2000*.*

We adjust the supply*a* 2 and the demand*b* 2 in the transportation solution tableau. Since the demand of the destination*D* 2 becomes zero, column2is eliminated from further consideration.

Transportation costs tableau Transportation solution tableau

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | *D*1 | *D*2 | *D*3 | Supply |
| *O*1 | 8 | 6 | 10 | 2000 |
| *O*2 | 10 | 4 | 9 | 2500 |
| Deman. | 1500 | 2000 | 1000 |  |

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | *D*1 | *D*2 | *D*3 | Supply |
| *O*1 |  |  |  | 2000 |
| *O*2 |  | 2000 |  | 500 |
| Deman. | 1500 | 0 | 1000 |  |

* + - * **Step 3**. More than one row or one column remain under consideration in the solution tableau. Therefore, we go to Step 1, and proceed with a new iteration of the algorithm.

### Second iteration.

Once again, we proceed to compute the row and column differences,*RD i* and *CDj*, for rows, columns and costs*c ij* still under consideration in the transportation costs tableau. In this case,*RD* 1 =*CD* 1 = 2have the same largest difference. The choice can be made randomly; we choose row 1, for instance. The smallest unit transportation cost in row 1 is*c* 11 = 8, which leads us to choose cell(1*,*1). We assign to the variable*x* 11 the maximum feasible amount consistent with the

row and the column requirements of cell(1*,*1):*x* 11 = min{1500*,*2000}= 1500. We adjust the supply*a* 1 and the demand*b* 1 in the transportation solution tableau. Since the demand of the destination*D* 1 becomes zero, column1is eliminated from further consideration.

Transportation costs tableau Transportation solution tableau

*RDi*

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | *D*1 | *D*2 | *D*3 | Supply |
| *O*1 | 8 | 6 | 10 | 2000 |
| *O*2 | 10 | 4 | 9 | 2500 |
| Deman. | 1500 | 2000 | 1000 |  |

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | *D*1 | *D*2 | *D*3 | Supply |
| *O*1 | 1500 |  |  | 500 |
| *O*2 |  | 2000 |  | 500 |
| Deman. | 0 | 0 | 1000 |  |

2

1

*CDj* 2 1

Only one column remains under consideration; the third one, which corre- sponds to the destination*D* 3. Thus, the two remaining cells(1*,*3)and(2*,*3)are selected and the remaining supplies are assigned to the variables associated with them:*x* 13 = 500and*x* 23 = 500. The algorithm stops; the solution tableau shows an initial basic feasible solution to the problem.

Transportation costs tableau Transportation solution tableau

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | *D*1 | *D*2 | *D*3 | Supply |
| *O*1 | 8 | 6 | 10 | 2000 |
| *O*2 | 10 | 4 | 9 | 2500 |
| Deman. | 1500 | 2000 | 1000 |  |

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | *D*1 | *D*2 | *D*3 | Supply |
| *O*1 | 1500 |  | 500 | 2000 |
| *O*2 |  | 2000 | 500 | 2500 |
| Deman. | 1500 | 2000 | 1000 |  |

* The solution.

*x*11 = 1500*, x*12 = 0*, x*13 = 500*, x*21 = 0*, x*22 = 2000*, x*23 = 500*.*

* The total transportation cost.

*z*= (8×1500) + (10×500) + (4×2000) + (9×500) = 29500*.*

This initial solution is better than the one found by applying the Northwest Corner method, because the total transportation cost is lower.�

# Least Cost Method

**Definition:** The **Least Cost Method** is another method used to obtain the initial feasible solution for the transportation problem. Here, the allocation begins with the cell which has the minimum cost. The lower cost cells are chosen over the higher-cost cell with the objective to have the least cost of transportation.

**The Least Cost Method is considered to produce more optimal results than the North-west Corner because it considers the shipping cost while making the allocation, whereas the North-West corner method only considers the availability and supply requirement and allocation begin with the extreme left corner, irrespective of the shipping cost.**

**Let’s understand the concept of Least Cost method through a problem given below:**

****

**In the given matrix, the supply of each source A, B, C is given Viz. 50units, 40 units, and 60 units respectively. The weekly demand for three retailers D, E, F i.e. 20 units, 95 units and 35 units is given respectively. The shipping cost is given for all the routes.**

**The minimum transportation cost can be obtained by following the steps given below:**

****

1. **The minimum cost in the matrix is Rs 3, but there is a tie in the cell BF, and CD, now the question arises in which cell we shall allocate. Generally, the cost where maximum quantity can be assigned should be chosen to obtain the better initial solution. Therefore, 35 units shall be assigned to the cell BF. With this, the demand for retailer F gets fulfilled, and only 5 units are left with the source B.**
2. **Again the minimum cost in the matrix is Rs 3. Therefore, 20 units shall be assigned to the cell CD. With this, the demand of retailer D gets fulfilled. Only 40 units are left with the source C.**
3. **The next minimum cost is Rs 4, but however, the demand for F is completed, we will move to the next minimum cost which is 5. Again, the demand of D is completed. The next minimum cost is 6, and there is a tie between three cells. But however, no units can be assigned to the cells BD and CF as the demand for both the retailers D and F are saturated. So, we shall assign 5 units to Cell BE. With this, the supply of source B gets saturated.**
4. **The next minimum cost is 8, assign 50 units to the cell AE. The supply of source A gets saturated.**
5. **The next minimum cost is Rs 9; we shall assign 40 units to the cell CE. With his both the demand and supply of all the sources and origins gets saturated.**

**The total cost can be calculated by multiplying the assigned quantity with the concerned cost of the cell. Therefore,**

**Total Cost = 50\*8 + 5\*6 + 35\*3 +20\*3 +40\*9 = Rs 955.**

**Note: The supply and demand should be equal and in case supply are more, the dummy source is added in the table with demand being equal to the difference between supply and demand, and the cost remains zero. Similarly, in case the demand is more than supply, then dummy destination or origin is added to the table with the supply equal to the difference in quantity demanded and supplied and the cost being zero.**

# Modified Distribution Method

**Definition:** The **Modified Distribution Method** or **MODI**is an efficient method of checking the optimality of the initial feasible solution.

**The concept of MODI can be further comprehended through an illustration given below:**

1. **Initial basic feasible solution is given below:**
2. **Now, calculate the values of ui and vj by using the equation:
ui+vj = Cij
Substituting the value of u1 as 0
U1+V1 = C11, 0+V1 = 6 or V1 = 6
U1 +V2 = C12, 0+V2 = 4 or V2 = 4
U2+V2 = C22, U2+4 = 8 or U2 = 4
U3+ V2 = C32, U3+4 = 4 or U3 = 0
U3+V3 = C33, 0+V3 = 2 or V3 =2**
3. **Next step is to calculate the opportunity cost of the unoccupied cells (AF, BD, BF, CD) by using the following formula:
Cij – (ui+Vi)

AF = C13 – (U1+V3),  1- (0+2) = -1 or 1
BD = C21 – (U2+v1),   3- (4+6) = -7 or 7
BF = C23 – (U2+V3),   7- (4+2) = 1 or -1
CD = C31- (U3+V1),    4- (0+6) = -2 or 2**
4. **Choose the largest positive opportunity cost, which is 7 and draw a closed path, as shown in the matrix below. Start from the unoccupied cell and assign “+” or “–“sign alternatively. Therefore, The most favored cell is BD, assign as many units as possible.**
5. **The matrix below shows the maximum allocation to the cell BD, and that number of units are added to the cell with a positive sign and subtracted from the cell with a negative sign.**
6. **Again, repeat the steps from 1 to 4 i.e. find out the opportunity costs for each unoccupied cell and assign the maximum possible units to the cell having the largest opportunity cost. This process will go on until the optimum solution is reached.**

**The Modified distribution method is an improvement over the stepping stone method since; it can be applied more efficiently when a large number of sources and destinations are involved, which becomes quite difficult or tedious in case of stepping stone method.**

**Modified distribution method reduces the number of steps involved in the evaluation of empty cells, thereby minimizes the complexity and gives a straightforward computational scheme through which the opportunity cost of each empty cell can be determined**

# The assignment problem

The assignment problem is a special case of the transportation problem. It deals with assigning*n*origins (workers, for instance) to*n*destinations (jobs or ma- chines, for instance,) with the goal of determining the minimum cost assignment. Each origin must be assigned to one and only one destination, and each destination must have assigned one and only one origin.*c ij* represents the cost of assigning the origin*O i* to the destination*D j*,*i, j*= 1*, . . . , n*.

Decision variables are deﬁned like this:

*xij*

=  1if*O i* is assigned to*D j*

 0otherwise

A linear model in standard form for the assignment problem is given by the following:

*n n*

min*z*=

� � *cij xij*

*i*=1

subject to

*n*

*j*=1

� *xij* = 1*, i*= 1*, . . . , n*

*j*=1 *n*

� *xij* = 1*, j*= 1*, . . . , n*

*i*=1

*xij* = 0*,*1*, i, j*= 1*, . . . , n*

The ﬁrst*n*constraints ensure that each origin is assigned to one and only one destination; the next*n*constraints ensure that each destination is assigned to one and only one origin.

If the number of origins and destinations are not equal, the assignment prob- lem is unbalanced. In order to balance it, we can always add as many dummy

origins or dummy destinations as necessary. The assignment costs of the dummy origins or dummy destinations are zero.

The relevant data for any assignment problem can be summarized in a matrix format using a tableau called*the assignment costs tableau*.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | *D*1 | *D*2 | *. . .* | *Dn* |
| *O*1 | *c*11 | *c*12 | *. . .* | *c*1*n* |
| *O*1. | *c*21. | *c*22. . . | *. . .* | *c*2*n*. |

*On*

*cn*1 *cn*2

*. . .*

*cnn*

 The assignment costs tableau

**Hungarian assignment method**

The Hungarian method of assignment provides us with an efficient means of finding the optimal solution. The Hungarian method is based upon the following principles:

(i)     If a constant is added to every element of a row and/or column of the cost matrix of an assignment problem the resulting assignment problem has the same optimum solution as the original problem or vice versa.

(ii)   The solution having zero total cost is considered as optimum solution.

Hungarian method of assignment problem (minimization case) can be summarized in the following steps:

**Step I:**Subtract the minimum cost of each row of the cost (effectiveness) matrix from all the elements of the respective row so as to get first reduced matrix.

**Step II:** Similarly subtract the minimum cost of each column of the cost matrix from all the elements of the respective column of the first reduced matrix. This is first modified matrix.

**Step III:**   Starting with row 1 of the first modified matrix, examine the rows one by one until a row containing exactly single zero elements is found. Make any assignment by making that zero in or enclose the zero inside a. Then cross (X) all  other zeros in the column in which the assignment was made. This eliminates the possibility of making further assignments in that column.

**Step IV:**When the set of rows have been completely examined, an identical procedure is applied successively to columns that is examine columns one by one until a column containing exactly single zero element is found. Then make an experimental assignment in that position and cross other zeros in the row in which the assignment has been made.

**Step V:** Continue these successive operations on rows and columns until all zeros have been either assigned or crossed out and there is exactly one assignment in each row and in each column. In such case optimal assignment for the given problem is obtained.

**Step VI:** There may be some rows (or columns) without assignment i.e. the total number of marked zeros is less than the order of the matrix. In such case proceed to step VII.

**Step VII:**Draw the least possible number of horizontal and vertical lines to cover all zeros of the starting table. This can be done as follows:

1.      Mark (√) in the rows in which assignments has not been made.

2.      Mark column with (√) which have zeros in the marked rows.

3.      Mark rows with (√) which contains assignment in the marked column.

4.      Repeat 2 and 3 until the chain of marking is completed.

5.      Draw straight lines through marked columns.

6.      Draw straight lines through unmarked rows.

By this way we draw the minimum number of horizontal and vertical lines necessary to cover all zeros at least once. It should, however, be observed that in all n x n matrices less than n lines will cover the zeros only when there is no solution among them. Conversely, if the minimum number of lines is n, there is a solution.

**Step VIII:**In this step, we

1.    � Select the smallest element, say X, among all the not covered by any of the lines of the table; and

2.     Subtract this value X from all of the elements in the matrix not covered by lines and add X to all those elements that lie at the intersection of the horizontal and vertical lines, thus obtaining the second modified cost matrix.

**Step IX:**Repeat Steps IV, V and VI until we get the number of lines equal to the order of matrix I, till an optimum solution is attained.

**Step X:**We now have exactly one encircled zero in each row and each column of the cost matrix. The assignment schedule corresponding to these zeros is the optimum assignment. The above technique is explained by taking the following examples

**Example 1**

A plant manager has four subordinates, and four tasks to be performed. The subordinates differ in efficiency and the tasks differ in their intrinsic difficulty. This estimate of the times each man would take to perform each task is given in the effectiveness matrix below.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|   | I | II | III | IV |
| A | 8 | 26 | 17 | 11 |
| B | 13 | 28 | 4 | 26 |
| C | 38 | 19 | 18 | 15 |
| D | 19 | 26 | 24 | 10 |

How should the tasks be allocated, one to a man, so as to minimize the total man hours?

**Solution**

**Step I :**Subtracting the smallest element in each row from every element in that row, we get the first reduced matrix.

|  |  |  |  |
| --- | --- | --- | --- |
| 0 | 18 | 9 | 3 |
| 9 | 24 | 0 | 22 |
| 23 | 4 | 3 | 0 |
| 9 | 16 | 14 | 0 |

**Step II:**Next, we subtract the smallest element in each column from every element in that column; we get the second reduced matrix.

**Step III:**Now we test whether it is possible to make an assignment using only zero distances.

|  |  |  |  |
| --- | --- | --- | --- |
| 0 | 14 | 9 | 3 |
| 9 | 20 | 0 | 22 |
| 23 | 0 | 3 | 0 |
| 9 | 12 | 14 | 0 |

(a)    Starting with row 1 of the matrix, we examine rows one by one until a row containing exactly single zero elements are found. We make an experimental assignment (indicated by) to that cell. Then we cross all other zeros in the column in which the assignment was made.

(b) When the set of rows has been completely examined an identical procedure is applied successively to columns. Starting with Column 1, we examine columns until a column containing exactly one remaining zero is found. We make an experimental assignment in that position and cross other zeros in the row in which the assignment was made. It is found that no additional assignments are possible. Thus, we have the complete �Zero assignment�,

A � I, B � III, C � II, D � IV

The minimum total man hours are computed as

|  |  |
| --- | --- |
| Optimal assignment | Man hours |
| A � I | 8 |
| B � III | 4 |
| C � II | 19 |
| D � IV | 10 |
| **Total** | **41 hours** |