#  Dual of LPP

For every linear programming problem there is a corresponding linear programming problem called the **dual**. If the original problem is a maximization problem then the dual problem is minimization problem and if the original problem is a minimization problem then the dual problem is maximization problem. In either case the final table of the dual problem will contain both the solution to the dual problem and the solution to the original problem.

The solution of the dual problem is readily obtained from the original problem solution if the simplex method is used.

The formulation of the dual problem also sometimes referred as the concept of duality is helpful for the understanding of the linear programming. The variable of the dual problem is known as the dual variables or shadow price of the various resources. The dual problem is easier to solve than the original problem. The dual problem solution leads to the solution of the original problem and thus efficient computational techniques can be developed through the concept of duality. Finally, in the competitive strategy problem solution of both the original and dual problem is necessary to understand the complete problem.

# Dual Problem Formulation

If the original problem is in the standard form then the dual problem can be formulated using the following rules:

The number of constraints in the original problem is equal to the number of dual variables. The number of constraints in the dual problem is equal to the number of variables in the original problem.

 The original problem profit coefficients appear on the right hand side of the dual problem constraints.

If the original problem is a maximization problem then the dual problem is a minimization problem. Similarly, if the original problem is a minimization problem then the dual problem is a maximization problem.

 The original problem has less than or equal to (≤) type of constraints while the dual problem has

greater than or equal to (≥) type constraints.

The coefficients of the constraints of the original problem which appear from left to right are placed from top to bottom in the constraints of the dual problem and vice versa.

The Dual Linear Programming Problem is explained with the help of the following Example 4.1.

# Example 4.1

Consider the following product mix problem:

Three machine shops A, B, C produces three types of products X, Y, Z respectively. Each product involves operation of each of the machine shops. The time required for each operation on various products is given as follows:

# Machine Shops

**Products** A B C **Profit per unit**

|  |  |  |
| --- | --- | --- |
| 10 | 7 | 2 |
| 2 | 3 | 4 |
| 1 | 2 | 1 |

X $12

* + 1. $3
		2. $1

Available Hours 100 77 80

The available hours at the machine shops A, B, C are 100, 77, and 80 only. The profit per unit of products X, Y, and Z is $12, $3, and $1 respectively.

# Solution:

The formulation of Linear Programming (original problem) is as follows: Maximize

Subject to:

12x1 + 3x2 + x3

10x1 + 2x2 + x3 ≤ 100

7x1 + 3x2 + 2x3 ≤ 77

2x1 + 4x2 + x3 ≤ 80

x1, x2, x3 ≥ 0

We introduce the slack variables s4, s5 and s6 then the equalities becomes as:

Maximize

12x1 + 3x2 + x3

Subject to:

10x1 + 2x2 + x3 + s4 = 100

7x1 + 3x2 + 2x3 + s5 = 77

2x1 + 4x2 + x3 +s6 = 80

x1, x2, x3, s4, s5, s6 ≥ 0

Form the above equations, the first simplex table is obtained is as follows:

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| CB | Basic Variable | Cj XB | 12x1 | 3x2 | 1x3 | 0s4 | 0s5 | 0s6 |
| 0 | s4 | 100 | 10 | 2 | 1 | 1 | 0 | 0 |

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| 00 | s5 s6 | 7780 | 72 | 34 | 21 | 00 | 10 | 01 |
|  | zj-cj |  | -12 | -3 | -1 | 0 | 0 | 0 |

Table 1

Note that the basic variables are s4, s5 and s6. Therefore CB1 = 0, CB2 = 0, CB3 = 0.

1. The smallest negative element in the above table of z1 – c1 is -12. Hence, x1 becomes a basic variable in the next iteration.
2. Determine the minimum ratios

Min 100, 72, 80 = 10

10 7 2

Here the minimum value is s4, which is made as a non-basic variable.

1. The next Table 2 is calculated using the following rules:
	1. The revised basic variables are x1, s5, s6. Accordingly we make CB1=22, CB2=0 and CB3=0.
	2. Since x1 is the incoming variable we make x1 coefficient one by dividing each element of row 1 by 10. Thus the numerical value of the element corresponding to x2 is 2/10, corresponding to x3 is 1/10, corresponding to s4 is 1/10, corresponding to s5 is 0/10 and corresponding to s6 is 0/10 in Table 2.
	3. The incoming basic variable should only appear in the first row. So we multiply first row of Table 2 by 7 and subtract if from the second row of Table 1 element by element.

Thus,

The element corresponding to x1 in the second row of Table 2 is zero

The element corresponding to x2 is 3 – 7 \* 2 = 16

10 10

By using this way we get the elements of the second and the third row in Table 2. Similarly, the calculation of numerical values of basic variables in Table 2 is done.

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| CB | Basic | Cj | 22 | 6 | 2 | 0 | 0 | 0 |
|  | Variable | XB | x1 | x2 | x3 | s4 | s5 | s6 |
| 12 | x1 | 10 | 1 | 2/10 | 1/10 | 1/10 | 0 | 0 |
| 0 | s5 | 7 | 0 | 16/10 | 13/10 | -7/10 | 1 | 0 |
| 0 | s6 | 60 | 0 | 18/5 | 4/5 | -1/5 | 0 | 1 |
|  | zj-cj |  | 0 | -3/5 | 1/5 | 6/5 | 0 | 0 |

Table 2

1. z2-c2 = -3/5. So x2 becomes a basic variable in the next iteration.
2. Determine the minimum of the ratios

10, 7 , 60

Min 2 16 18 = Min 50, 70, 300 = 70

10 10 5 16 18 16

So that the variable s5 will be a non basic variable in the next iteration.

1. From Table 2, the Table 3 is calculated using the rules (i), (ii) and (iii) mentioned above.

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| CB | Basic | Cj | 12 | 3 | 1 | 0 | 0 | 0 |
|  | Variable | XB | x1 | x2 | x3 | s4 | s5 | s6 |
| 12 | x1 | 73/8 | 1 | 0 | -1/16 | 3/16 | -1/8 | 0 |
| 3 | s5 | 35/8 | 0 | 1 | 13/16 | -7/16 | 5/8 | 0 |
| 0 | s6 | 177/4 | 0 | 0 | -17/8 | 11/8 | -9/4 | 1 |
|  | zj-cj |  | 0 | 0 | 11/16 | 15/16 | 3/8 | 0 |

Table 3

Since all the zi – cj ≥ 0, the optimum solution is as:

x1 = 73/8 and x2 = 35/8 and The Maximum Profit is: $981/8 = $122.625

Suppose an investor is deciding to purchase the resources A, B, C. What offers is he going to produce?

Let, assume that W1, W2 and W3 are the offers made per hour of machine time A, B and C respectively. Then these prices W1, W2 and W3 must satisfy the conditions given below:

1. W1, W2, W3 ≥ 0

1. Assume that the investor is behaving in a rational manner; he would try to bargain as much as possible so that the total annual payable to the produces would be as little as possible. This leads to the following condition:

Minimize

100W1 + 77W2 + 80W3

1. The total amount offer by the investor to the three resources viz. A, B and C required to produce one unit of each product must be at least as high as the profit gained by the producer per unit.

Since, these resources enable the producer to earn the specified profit corresponding to the product he would not like to sell it for anything less assuming he is behaving rationally. This leads to the following conditions:

10w1 + 7w2 + 2w3 ≥ 12

2w1 + 3w2 + 4w3 ≥ 3

w1 + 2w2 + w3 ≥ 1

Thus, in this case we have a linear problem to ascertain the values of the variable w1, w2, w3. The variables w1, w2 and w3 are called as **dual variables**.

# Note:

The original (primal) problem illustrated in this example

1. considers the objective function maximization
2. contains ≤ type constraints
3. has non-negative constraints

This original problem is called as primal problem in the standard form.

# Dual Problem Properties

The following are the different properties of dual programming problem:

1. If the original problem is in the standard form, then the dual problem solution is obtained from the zj – cj values of slack variables.

For example: In the Example 4.1, the variables s4, s4 and s6 are the slack variables. Hence the dual problem solution is w1 = z4 – c4 = 15/16, w2 = z5 – c5 = 3/8 and w3 = z6 – c6 = 0.

1. The original problem objective function maximum value is the minimum value of the dual problem objective function.

For example:

From the above Example 4.1 we know that the original problem maximum values is 981/8 = 122.625. So that the minimum value of the dual problem objective function is

100\*15/16 + 77\*3/8 + 80\*0 = 981/8

Here the result has an important practical implication. If both producer and investor analyzed the problem then neither of the two can outmaneuver the other.

1. Shadow Price: A resource shadow price is its unit cost, which is equal to the increase in profit to be realized by one additional unit of the resource.

For example:

Let the minimum objective function value is expressed as:

100\*15/16 + 77\*3/8 + 80\*0

If the first resource is increased by one unit the maximum profit also increases by 15/16, which is the first dual variable of the optimum solution. Therefore, the dual variables are also referred as the resource shadow price or imputed price. Note that in the previous example the shadow price of the third resource is zero because there is already an unutilized amount, so that profit is not increased by more of it until the current supply is totally exhausted.

1. In the originals problem, if the number of constraints and variables is m and n then the constraint and variables in the dual problem is n and m respectively. Suppose the slack variables in the original problem is represented by y1, y1, ….., yn and the surplus variables are represented by z1, z2, …, zn in the dual problem.
2. Suppose, the original problem is not in a standard form, then the dual problem structure is unchanged. However, if a constraint is greater than or equal to type, the corresponding dual variable is negative or zero. Similarly, if a constraint in the original problem is equal to type, then the corresponding dual variables is unrestricted in sign.

# Example 4.2

Consider the following linear programming problem Maximize

22x1 + 25x2 +19x3

Subject to:

18x1 + 26x2 + 22x3 ≤ 350

14x1 + 18x2 + 20x3 ≥180

17x1 + 19x2 + 18x3 = 205

x1, x2, x3 ≥ 0

Note that this is a primal or original problem.

The corresponding dual problem for this problem is as follows: Minimize

250w1 + 80w2 +105w3

Subject to:

18w1 + 4w2 + 7w3 ≥ 22

26w1 + 18w2 + 19w3 ≥ 25

22w1 + 20w2 + 18w3 ≥ 19

w1 ≥ 0, w2 ≤, and w3 is unrestricted in sign (+ or -).

Now, we can solve this using simplex method as usual.

# Simple Way of Solving Dual Problem

Solving of dual problem is simple; this is illustrated with the help of the following Example 4.3.

# Example 4.3:

Minimize

P = x1 + 2x2

# Solution:

Subject to:

x1 + x2 ≥ 8 2x1 + y ≥ 12 x1 ≥ 1

Step 1: Set up the P-matrix and its transpose

1 1 8

2 1 12

P = 1 0 1

1 2 0

1 2 1 1

PT = 1 1 0 2

8 12 1 0

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| w1 | w2 | w3 | s1 |  | s2 |  | g |  | z |
| 1 | 2 | 1 | 1 | 0 | 0 | 1 |
| 1 | 1 | 0 | 0 | 1 | 0 | 2 |
| -8 | -12 | -1 | 0 | 0 | 1 | 0 |

Step 2: Form the objective function

constraints and the for the dual

w1 + 2w2 + w3 ≤ 1

w1 + w2 ≤ 2

z = 8w1 = 12w2 + 2

Step 3: Construct the initial simplex tableau for the dual

Since there are no negative entries in the last column above the third row, we have a standard simplex problem. The most negative number in the bottom row to the left of the last column is

**-12**. This establishes the pivot column. The smallest nonnegative ratio is 1/2. The pivot element is **2** in the w2-column.

Step 4: Pivoting

Pivoting about the **2** we get:

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| w1 | w2 | w3 | s1 | s2 |  | g |  | z |
| 1 | 2 | 1 | 1 | 0 | 0 | 1 |
| 1/2 | 0 | -1/2 | -1/2 | 1 | 0 | 3/2 |
| -2 | 0 | 5 | 6 | 0 | 1 | 6 |

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| w1 | w2 | w3 | s1 |  | s2 |  | g |  | z |
| 1/2 | **1** | 1/2 | 1/2 | 0 | 0 | 1/2 |
| 1 | 1 | 0 | 0 | 1 | 0 | 2 |
| -8 | -12 | -1 | 0 | 0 | 1 | 0 |

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| w1 | w2 | w3 | s1 | s2 |  | g |  | z |
| 1/2 | **1** | 1/2 | 1/2 | 0 | 0 | 1/2 |
| 1/2 | 0 | -1/2 | -1/2 | 1 | 0 | 3/2 |
| -2 | 0 | 5 | 6 | 0 | 1 | 6 |

The most negative entry in the bottom row to the left of the last column is **-2**. The smallest non- negative ratio is the **1/2** in the first row. This is the next pivot element.

Pivoting about the **1/2**:

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| w1 | w2 | w3 | s1 | s2 |  | g |  | z |
| **1/2** | 1 | 1/2 | 1/2 | 0 | 0 | 1/2 |
| 1/2 | 0 | -1/2 | -1/2 | 1 | 0 | 3/2 |
| -2 | 0 | 5 | 6 | 0 | 1 | 6 |

Since there are no negative entries in the bottom row and to the left of the last column, the process is complete. The solutions are at the feet of the slack variable columns.

Therefore,

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| w1 | w2 | w3 | s1 | s2 | g | z |
| **1** | 2 | 1 | 1 | 0 | 0 | 1 |
| 0 | -1 | -1 | -1 | 1 | 0 | 1 |
| 0 | 4 | 7 | 8 | 0 | 1 | 8 |

The optimum solution

provided by x1 = 8 and x2 = 0

The Minimum Value

is: 8

# Summary

For every linear programming problem there is a dual problem. The variables of the dual problem are called as dual variables. The variables have economic value, which can be used for planning its resources. The dual problem solution is achieved by the simplex method calculation of the original (primal) problem. The dual problem solution has certain properties, which may be very useful for calculation purposes.

# Key Terms

**Original Problem**: This is the original linear programming problem, also called as primal problem.

**Dual Problem**: A dual problem is a linear programming problem is another linear programming problem formulated from the parameters of the primal problem.

**Dual Variables**: Dual programming problem variables.

**Optimum Solution**: The solution where the objective function is minimized or maximized.

**Shadow Price**: Price of a resource is the change in the optimum value of the objective function per unit increase of the resource.