UNIT- 4

FOURIER SERIES SERIES

# 4.1 FOURIER SERIES FOR PERIODIC FUNCTIONS

This section explains three Fourier series: sines, cosines, and exponentials *eikx*. Square waves (1 or 0 or 1) are great examples, with delta functions in the derivative. We look at a spike, a step function, and a ramp—and smoother functions too.

−

Start with sin *x*. It has period 2*π* since sin(*x* + 2*π*) = sin *x*. It is an odd function since sin( *x*) = sin *x*, and it vanishes at *x* = 0 and *x* = *π*. Every function sin *nx* has those three properties, and Fourier looked at *inﬁnite combinations of the sines*:

Fourier sine series *S*(*x*) = *b* sin *x* + *b* sin 2*x* + *b* sin 3*x* + ··· = *b* sin *nx* (1)

1

2

3

Σ

∞

*n*

*n*=1

− −

If the numbers *b*1*, b*2*,...* drop oﬀ quickly enough (we are foreshadowing the im- portance of the decay rate) then the sum *S*(*x*) will inherit all three properties:

Periodic *S*(*x* + 2*π*) = *S*(*x*) Odd *S*(−*x*) = −*S*(*x*) *S*(0) = *S*(*π*) = 0

200 years ago, Fourier startled the mathematicians in France by suggesting that *any function S*(*x*) with those properties could be expressed as an inﬁnite series of sines. This idea started an enormous development of Fourier series. Our ﬁrst step is to compute from *S*(*x*) the number *bk* that multiplies sin *kx*.

*Suppose S*(*x*) = Σ *bn* sin *nx*. *Multiply both sides by* sin *kx*. *Integrate from* 0 *to π*:

∫

∫

∫

π

*S*(*x*) sin *kxdx* =

0

π

*b*1 sin *x* sin *kx dx* + ··· +

0

π

*bk* sin *kx* sin *kx dx* + ··· (2)

0

On the right side, all integrals are zero except the highlighted one with *n* = *k*. This property of “orthogonality” will dominate the whole chapter. The sines make 90◦ angles in function space, when their inner products are integrals from 0 to *π*:

Orthogonality

∫

π

sin *nx* sin *kxdx* = 0 if *n* ƒ= *k.*

(3)

0

Zero comes quickly if we integrate ∫ cos *mx dx* = Σ sin *mx* Σ*π* = 0 − 0. So we use this:

*m*

0

1

1

 

Product of sines sin *nx* sin *kx* = 2 cos(*n* − *k*)*x* − 2 cos(*n* + *k*)*x.* (4)

Integrating cos *mx* with *m* = *n* − *k* and *m* = *n* + *k* proves orthogonality of the sines.

*The exception is when n* = *k*. Then we are integrating (sin *kx*)2 = 1 − 1 cos 2*kx*:

∫ π ∫ π 1

sin *kx* sin *kxdx* =



0

0

2 2

∫ π 1 *π*

2 *dx* −

cos 2*kxdx* = *.* (5) 2 2

 

0

The highlighted term in equation (2) is *bkπ/*2. Multiply both sides of (2) by 2*/π*:



Sine coeﬃcients

*S*(−*x*) = −*S*(*x*)

*bk* =

2

*π*

∫

π

1

*S*(*x*) sin *kxdx* =

*π*

∫

π

*S*(*x*) sin *kxdx.*

(6)

0

−π

Notice that *S*(*x*) sin *kx* is *even* (equal integrals from −*π* to 0 and from 0 to *π*).

I will go immediately to the most important example of a Fourier sine series. *S*(*x*) is an odd square wave with *SW* (*x*) = 1 for 0 *< x < π*. It is drawn in Figure 4.1 as an odd function (with period 2*π*) that vanishes at *x* = 0 and *x* = *π*.

*SW* (*x*) = 1



−*π*

0

*π*

2*π*

*x*

Figure 4.1: The odd square wave with *SW* (*x* + 2*π*) = *SW* (*x*) = {1 or 0 or −1}.

Example 1 Find the Fourier sine coeﬃcients *bk* of the square wave *SW* (*x*).

Solution For *k* = 1*,* 2*,...* use the ﬁrst formula (6) with *S*(*x*) = 1 between 0 and *π*:

2 ∫ π

*bk* = *π*

sin *kxdx* =

*π*

=

*π*

*, , , , , ,...*

1 2 3 4 5 6

(7)



0

2 Σ− cos *kx* Σπ

 

*k*

0

2 . 2 0 2 0 2 0 Σ

      

The even-numbered coeﬃcients *b*2*k* are all zero because cos 2*kπ* = cos 0 = 1. The odd-numbered coeﬃcients *bk* = 4*/πk* decrease at the rate 1*/k*. We will see that same 1*/k* decay rate for all functions formed from *smooth pieces and jumps*.

Put those coeﬃcients 4*/πk* and zero into the Fourier sine series for *SW* (*x*):

Square wave *SW* (*x*) = 4 Σ sin *x* + sin 3*x* + sin 5*x* + sin 7*x* + ··· Σ (8)

*π*

1

3

5

7

Figure 4.2 graphs this sum after one term, then two terms, and then ﬁve terms. You can see the all-important Gibbs phenomenon appearing as these “partial sums”

include more terms. Away from the jumps, we safely approach *SW* (*x*) = 1 or 1. At *x* = *π/*2, the series gives a beautiful alternating formula for the number *π*:

−

1 = 4 Σ 1 − 1 + 1 − 1 + ··· Σ so that *π* = 4Σ 1 − 1 + 1 − 1 + ·· ·Σ*.* (9)

*π*

1

3

5

7

1

3

5

7

*The Gibbs phenomenon is the overshoot that moves closer and closer to the jumps*. Its height approaches 1*.*18 *...* and it does not decrease with more terms of the series! Overshoot is the one greatest obstacle to calculation of all discontinuous functions (like shock waves in ﬂuid ﬂow). We try hard to avoid Gibbs but sometimes we can’t.

. Σ . Σ

4 sin *x* sin 3*x* 4 sin *x* sin 9*x* Solid curve + 5 terms: + ··· +

*π*

1

3

*π*

1

9

4 sin *x*



Dashed

overshoot−→ *SW* = 1

*π* 1

*π π x π x*

−

2

Figure 4.2: Gibbs phenomenon: Partial sums Σ*N bn* sin *nx* overshoot near jumps.

1

# Fourier Coeﬃcients are Best

Let me look again at the ﬁrst term *b*1 sin *x* = (4*/π*) sin *x*. This is the closest possible approximation to the square wave *SW* , by any multiple of sin *x* (closest in the least squares sense). To see this optimal property of the Fourier coeﬃcients, minimize the error over all *b*1:

The error is

*π*

(*SW* −*b*1 sin *x*)2 *dx* The *b*1 derivative is −2

∫

0

*π*

(*SW b*1 sin *x*) sin *xdx.*

∫

−

0

0

The integral of sin2 *x* is *π/*2. So the derivative is zero when *b*1

This is exactly equation (6) for the Fourier coeﬃcient.

= (2*/π*) ∫ *π S*(*x*) sin *xdx*.

Each *bk* sin *kx* is as close as possible to *SW* (*x*). We can ﬁnd the coeﬃcients *bk* one at a time, *because the sines are orthogonal*. The square wave has *b*2 = 0 because all other multiples of sin 2*x* increase the error. Term by term, we are “projecting the function onto each axis sin *kx*.”

# Fourier Cosine Series

The cosine series applies to *even functions* with *C*(−*x*) = *C*(*x*):

∞

Σ

Cosine series *C*(*x*) = *a*0 + *a*1 cos *x* + *a*2 cos 2*x* + ··· = *a*0 + *an* cos *nx.* (10)

*n*=1

Every cosine has period 2*π*. Figure 4.3 shows two even functions, the repeating ramp *RR*(*x*) and the up-down train *UD*(*x*) of delta functions. That sawtooth ramp *RR* is the integral of the square wave. The delta functions in *UD* give the derivative of the square wave. (For sines, the integral and derivative are cosines.) *RR* and *UD* will be valuable examples, one smoother than *SW* , one less smooth.

First we ﬁnd formulas for the cosine coeﬃcients *a*0 and *ak*. The constant term *a*0

is the *average value* of the function *C*(*x*):

1 π

∫

*a*0 = Average *a*0 =

*π*

0

1

*C*(*x*) *dx* =

2*π*

π

*C*(*x*) *dx.* (11)

∫

−π

I just integrated every term in the cosine series (10) from 0 to *π*. On the right side, the integral of *a*0 is *a*0*π* (divide both sides by *π*). All other integrals are zero:

π

∫

cos *nx dx* =

0

sin *nx* π



*n* 0

Σ Σ

= 0 − 0 = 0*.* (12)

In words, the constant function 1 is orthogonal to cos *nx* over the interval [0*, π*].

The other cosine coeﬃcients *ak* come from the *orthogonality of cosines*. As with sines, we multiply both sides of (10) by cos *kx* and integrate from 0 to *π*:

π π

∫

∫

*C*(*x*) cos *kxdx* =

0 0

*a*0 cos *kxdx*+

π

*a*1 cos *x* cos *kxdx*+··+

∫

0

π

*ak*(cos *kx*)2

∫

0

*dx*+··

You know what is coming. On the right side, only the highlighted term can be nonzero. Problem 4.1.1 proves this by an identity for cos *nx* cos *kx*—now (4) has a plus sign. The bold nonzero term is *akπ/*2 and we multiply both sides by 2*/π*:



Cosine coeﬃcients

*C*(−*x*) = *C*(*x*)

*ak* =

2

*π*

∫

π

1

*C*(*x*) cos *kxdx* =

*π*

∫

π

*C*(*x*) cos *kxdx.* (13)

0

−π

Again the integral over a full period from −*π* to *π* (also 0 to 2*π*) is just doubled.

*x x*



*RR*(*x*)=|*x*|



2*δ*(*x*) ^

2*δ*(*x* − 2*π*) ^

Up-down *UD*(*x*)

−*π*

0

*π*

2*π*

7−2*δ*(*x* + *π*)

7−2*δ*(*x* − *π*)

−*π* 0 *π* 2*π*

Repeating Ramp *RR*(*x*)

Integral of Square Wave

Figure 4.3: The repeating ramp *RR* and the up-down *UD* (periodic spikes) are even. The derivative of *RR* is the odd square wave *SW* . The derivative of *SW* is *UD*.

Example 2 Find the cosine coeﬃcients of the ramp *RR*(*x*) and the up-down *UD*(*x*).

Solution The simplest way is to start with the sine series for the square wave:

*SW* (*x*) = 4 Σ sin *x* + sin 3*x* + sin 5*x* + sin 7*x* + ··· Σ *.*

*π*

1

3

5

7

Take the derivative of every term to produce cosines in the up-down delta function:

4

Up-down series *UD*(*x*) = [cos *x* + cos 3*x* + cos 5*x* + cos 7*x* + ··· ] *.* (14)

*π*

Those coeﬃcients don’t decay at all. The terms in the series don’t approach zero, so oﬃcially the series cannot converge. Nevertheless it is somehow correct and important. Unoﬃcially this sum of cosines has all 1’s at *x* = 0 and all −1’s at *x* = *π*. Then +∞ and −∞ ar∫e consistent with 2*δ*(*x*) and −2*δ*(*x* − *π*). The true way to recognize *δ*(*x*) is

by the test

*δ*(*x*)*f* (*x*) *dx* = *f* (0) and Example 3 will do this.

For the repeating ramp, we integrate the square wave series for *SW* (*x*) and add the average ramp height *a*0 = *π/*2, halfway from 0 to *π*:

Ramp series *RR*(*x*) = *π* − *π* Σ cos *x* + cos 3*x* + cos 5*x* + cos 7*x* + ··· Σ *.* (15)

2

4

12

32

52

72

The constant of integration is *a*0. *Those coeﬃcients ak drop oﬀ like* 1*/k*2. They could be computed directly from formula (13) using *x* cos *kxdx*, but this requires an integration by parts (or a table of integrals or an appeal to *Mathematica* or *Maple*). It was much easier to integrate every sine separately in *SW* (*x*), which makes clear the crucial point:

∫

Each “degree of smoothness” in the function is reﬂected in a faster decay rate of its Fourier coeﬃcients *ak* and *bk*.

No decay Delta functions (with spikes)

1*/k* decay Step functions (with jumps)

1*/k*2 decay Ramp functions (with corners)

1*/k*4 decay Spline functions (jumps in *f* jjj)

*rk* decay with *r <* 1 Analytic functions like 1*/*(2 − cos *x*)

Each integration divides the *k*th coeﬃcient by *k*. So the decay rate has an extra 1*/k*. The “Riemann-Lebesgue lemma” says that *ak* and *bk* approach zero for any continuous function (in fact whenever |*f* (*x*)|*dx* is ﬁnite). Analytic functions achieve a new level of smoothness—they can be diﬀerentiated forever. Their Fourier series and Taylor series in Chapter 5 converge exponentially fast.

∫

The poles of 1*/*(2 cos *x*) will be complex solutions of cos *x* = 2. Its Fourier series converges quickly because *rk* decays faster than any power 1*/kp*. Analytic functions are ideal for computations—the Gibbs phenomenon will never appear.

−

Now we go back to *δ*(*x*) for what could be the most important example of all.

Example 3 Find the (cosine) coeﬃcients of the *delta function δ*(*x*), made 2*π*-periodic.

Solution The spike occurs at the start of the interval [0*, π*] so safer to integrate from

−*π* to *π*. We ﬁnd *a*0 = 1*/*2*π* and the other *ak* = 1*/π* (cosines because *δ*(*x*) is even):

1 ∫ π 1 1 ∫ π 1

−π

Average *a*0 = 2*π*

*δ*(*x*) *dx* = 2*π* Cosines *ak* = *π*

*δ*(*x*) cos *kxdx* =

−π *π*

Then the series for the delta function has all cosines in equal amounts:



Delta function

*δ*(*x*) = + [cos *x* + cos 2*x* + cos 3*x* + ··· ] *.*

1 1

2*π π*

(16)

Again this series cannot truly converge (its terms don’t approach zero). But we can graph the sum after cos 5*x* and after cos 10*x*. Figure 4.4 shows how these “partial sums” are doing their best to approach *δ*(*x*). They oscillate faster and faster away from *x* = 0.

Actually there is a neat formula for the partial sum *δN* (*x*) that stops at cos *Nx*. Start by writing each term 2 cos *θ* as *eiθ* + *e*−*iθ*:

1

*δN* =

2*π*

1

[1 + 2 cos *x* + ··· +2 cos *Nx*] = 2*π*

Σ1+ *eix* + *e*−*ix* + ··· + *eiNx* + *e*−*iNx*Σ *.*

This is a geometric progression that starts from *e*−*iNx* and ends at *eiNx*. We have powers of the same factor *eix*. The sum of a geometric series is known:

1

*i*(*N* + 1 )*x*



−*i*(*N* + )*x* 1





Partial sum

up to cos *Nx*

1 *e*

*δN* (*x*) = 2*π*

2 − *e*

*eix/*2 − *e*−*ix/*2

2 = 1 sin(*N* + 2 )*x.* (17)

2*π* sin 1 *x*

2

This is the function graphed in Figure 4.4. We claim that for any *N* the area underneath *δN* (*x*) is 1. (Each cosine integrated from *π* to *π* gives zero. The integral of 1*/*2*π* is 1.) The central “lobe” in the graph ends when sin(*N* + 1 )*x* comes down to zero, and that happens when (*N* + 1 )*x* = *π*. I think the area under that lobe (marked by bullets) approaches the same number 1*.*18 *...* that appears in the Gibbs phenomenon.

−

2

2

±

In what way does *δN* (*x*) approach *δ*(*x*)? The terms cos *nx* in the series jump around at each point *x* = 0, not approaching zero. At *x* = *π* we see 1 [1 − 2+2 − 2+ ··· ] and

2*π*

the sum is 1*/*2*π* or −1*/*2*π*. The bumps in the partial sums don’t get smallerΣthan 1*/*2*π*.

and integrate, because *we only know δ*(*x*) *from its integrals* ∫ *δ*(*x*)*f* (*x*) *dx* = *f* (0):

The right test for the delta function *δ*(*x*) is to multiply by a smooth *f* (*x*) =

*ak* cos *kx*

Weak convergence ∫ *π*

of *δ*

*N*

(*x*) to *δ*(*x*)

−*π*

*δN*(*x*)*f* (*x*) *dx* = *a*0 + ·· · + *aN* → *f* (0) *.* (18)

In this integrated sense (*weak sense*) the sums *δN* (*x*) do approach the delta function ! The convergenceΣof *a*0 + ··· + *aN* is the statement that at *x* = 0 the Fourier series of a

smooth *f* (*x*) =

*ak* cos *kx* converges to the number *f* (0).

*δ*10(*x*) height 21*/*2*π*

δ5(x) height 11/2π

−*π* 0

height 1*/*2*π*

*π* height −1/2π

Figure 4.4: The sums *δN* (*x*) = (1 + 2 cos *x* + ··· +2 cos *Nx*)*/*2*π* try to approach *δ*(*x*).

# Complete Series: Sines and Cosines

Over the half-period [0*, π*], the sines are not orthogonal to all the cosines. In fact the integral of sin *x* times 1 is not zero. So for functions *F* (*x*) that are not odd or even, we move to the complete series (sines plus cosines) on the full interval. Since our functions are periodic, that “full interval” can be [−*π, π*] or [0*,* 2*π*]:

Complete Fourier series

*F* (*x*) = *a* + *a* cos *nx* + *b* sin *nx .*

0

∞

Σ

*n*

Σ

∞

*n*

(19)

*n*=1

*n*=1

On every “2*π* interval” all sines and cosines are mutually orthogonal. We ﬁnd the

Fourier coeﬃcients *ak* and *bk* in the usual way: Multiply (19) by 1 and cos *kx* and

sin *kx*, and integrate both sides from −*π* to *π*:

1

a0 = 2*π*

π 1 π

*F* (*x*) *dx* ak = *π*

∫

∫

−π −π

1 π

*F* (*x*) cos *kxdx* bk = *π*

∫

−π

*F* (*x*) sin *kxdx.* (20)

Orthogonality kills oﬀ inﬁnitely many integrals and leaves only the one we want.

Another approach is to split *F* (*x*) = *C*(*x*) + *S*(*x*) into an even part and an odd part. Then we can use the earlier cosine and sine formulas. The two parts are

*C*(*x*) = *F*

even

(*x*) = *F* (*x*)+ *F* (−*x*)

2

*S*(*x*) = *F*

odd

(*x*) = *F* (*x*) − *F* (−*x*) *.* (21)

2

The even part gives the *a*’s and the odd part gives the *b*’s. Test on a short square pulse from *x* = 0 to *x* = *h*—this one-sided function is not odd or even.

Example 4 Find the *a*’s and *b*’s if *F* (*x*) = square pulse = 1 for 0 *< x < h*

.

0 for *h < x <* 2*π*

Solution The integrals for *a*0 and *ak* and *bk* stop at *x* = *h* where *F* (*x*) drops to zero. The coeﬃcients decay like 1*/k* because of the jump at *x* = 0 and the drop at *x* = *h*:

1

Coeﬃcients of square pulse *a*0 = 2*π*

*h h*

1 *dx* = = average

∫

0 2*π*

*πk bk* = *π*

1 ∫ *h*

*ak* = *π*

cos *kxdx* =

sin *kxdx* =

*.* (22)

*πk*



0

sin *kh*



1 ∫ *h*



0

1 − cos *kh*



If we divide *F* (*x*) by *h*, its graph is a tall thin rectangle: height 1 , base *h*, and area = 1.

*h*

When *h* approaches zero, *F* (*x*)*/h* is squeezed into a very thin interval. *The tall rectangle approaches (weakly) the delta function δ*(*x*). The average height is area*/*2*π* = 1*/*2*π*. Its other coeﬃcients *ak/h* and *bk/h* approach 1*/π* and 0, already known for *δ*(*x*):

*F* (*x*) *ak* 1 sin *kh* 1

→ *δ*(*x*) = →

and *bk* = 1 − cos *kh* → 0 as *h* → 0*.* (23)

*h h π kh π h πkh*

When the function has a jump, its Fourier series picks the halfway point. This example would converge to *F* (0) = 1 and *F* (*h*) = 1 , halfway up and halfway down.

2 2

The Fourier series converges to *F* (*x*) at each point where the function is smooth. This is a highly developed theory, and Carleson won the 2006 Abel Prize by proving convergence for every *x* except a set of measure zero. If the function has ﬁnite energy

∫

|*F* (*x*)|2 *dx*, he showed that the Fourier series converges “almost everywhere.”

# Energy in Function = Energy in Coeﬃcients

There is an extremely important equation (*the energy identity* ) that comes from integrating (*F* (*x*))2. When we square the Fourier series of *F* (*x*), and integrate from

2

−

*π* to *π*, all the “cross terms” drop out. The only nonzero integrals come from 1

and cos2 *kx* and sin2 *kx*, multiplied by *a*2 and *a*2 and *b*2:

0 *k k*

Energy in *F* (*x*) = (*a* + *a* cos *kx* + *b* sin *kx*) *dx*

∫

∫

π

−*π*

Σ

2

2 2

−*π*

(*F* (*x*)) *dx* = 2*πa* + *π*(*a* + *b* + *a* + *b* + ·· · ).

0

1

1

2 2

Σ

2

0

*k*

*k*

*π*

2

2

2

(24)

The energy in *F* (*x*) equals the energy in the coeﬃcients. The left side is like the

length squared of a vector, except *the vector is a function*. The right side comes from

an inﬁnitely long vector of *a*’s and *b*’s. The lengths are equal, which says that the

Fourier transf√orm from√function to vector is like an orthogonal matrix. Normalized

by constants 2*π* and *π*, we have an *orthonormal basis in function space*.

What is this function space ? It is like ordinary 3-dimensional space, except the “vectors∫” are functions. Their length ǁ*f* ǁ comes from integrating instead of adding:

2

ǁ*f* ǁ

2

=

|*f* (*x*)| *dx*. These functions ﬁll Hilbert space. The rules of geometry hold:

Length ǁ*f* ǁ2 = (*f, f* ) comes from the inner product (*f, g*) = *f* (*x*)*g*(*x*) *dx*

∫

Orthogonal functions (*f, g*) = 0 produce a right triangle: ǁ*f* + *g*ǁ2 = ǁ*f* ǁ2 + ǁ*g*ǁ2

I have tried to draw Hilbert space in Figure 4.5. It has inﬁnitely many axes. *The energy identity* (24) *is exactly the Pythagoras Law in inﬁnite-dimensional space*.

*v*2*k*−1 =

cos *kx*

√*π* <



sin *kx*

’ *v*2 =

sin *x*

√*π*

*v*2*k* = √*π*



7 *f* = *A*0*v*0 + *A*1*v*1 + *B*1*v*2 + ···

function in Hilbert space

2 2 2 2

90◦

1 L (*vi, vj*)= 0 €



ǁ*f* ǁ = *A*0 + *A*1 + *B*1 + ···

cos *x*

*v*0 = √2*π v*1 = √*π*

Figure 4.5: The Fourier series is a combination of orthonormal *v*’s (sines and cosines).

# Complex Exponentials ckeikx

This is a small step and we have to take it. In place of separate formulas for *a*0 and *ak*

and *bk*, we will have *one formula* for all the complex coeﬃcients *ck*. And the function

*F* (*x*) might be complex (as in quantum mechanics). The Discrete Fourier Transform will be much simpler when we use *N* complex exponentials for a vector. We practice in advance with the complex inﬁnite series for a 2*π*-periodic function:

Complex Fourier series *F* (*x*) = *c*0 + *c*1*eix* + *c*−1*e*−*ix* + ··· =

Σ

∞

*c e* (25)

*inx*

*n*

*n*=−∞

If every *cn* = *c*−*n*, we can combine *einx* with *e*−*inx* into 2 cos *nx*. Then (25) is the cosine series for an even function. If every *cn* = *c*−*n*, we use *einx e*−*inx* = 2*i* sin *nx*. Then (25) is the sine series for an odd function and the *c*’s are pure imaginary.

To ﬁnd *ck*, multiply (25) by *e*−*ikx* (not *eikx*) and integrate from −*π* to *π*:

− −

∫

∫

∫

∫

π

*F* (*x*)*e*−*ikx*

−π

*dx* =

π

*c*0*e*−*ikx*

−π

*dx*+

π

*c*1*eix*

−π

*e*−*ikx*

*dx*+···+

π

*ckeikx*

−π

*e*−*ikx*

*dx* +···

The complex exponentials are orthogonal. Every integral on the right side is zero, except for the highlighted term (when *n* = *k* and *eikxe*−*ikx* = 1). The integral of 1 is 2*π*. That surviving term gives the formula for *ck*:

Fourier coeﬃcients

∫

π

*F* (*x*)*e dx* = 2*πck*

−*ikx*

for *k* = 0*,* ±1*,...*

(26)

−π

Notice that *c*0 = *a*0 is still the average of *F* (*x*), because *e*0 = 1. The orthogonality of *einx* and *eikx* is checked by integrating, as always. But the complex inner product (*F, G*) takes the *complex conjugate G of G*. Before integrating, change *eikx* to *e*−*ikx*:

Complex inner product Orthogonality of *einx* and *eikx*

∫

∫

π

(*F, G*) =



*F* (*x*)*G*(*x*) *dx*

π

*ei*(*n*−*k*)*xdx* =

Σ *ei*(*n*−*k*)*x* Σπ

= 0 *.*

(27)

−π −π

*i*(*n* − *k*) −π

Example 5 Add the complex series for 1*/*(2 *eix*) and 1*/*(2 *e*−*ix*). These geometric series have exponentially fast decay from 1*/*2*k*. The functions are analytic.

− −

1 *eix*

.

+ +

2 4

*e*2*ix*

8 + ·· +

Σ

1 *e*−*ix*

+ +

.

2 4

*e*−2*ix*

8 + ··

Σ

= 1 +

cos *x*

+

2

cos 2*x*

+

4

cos 3*x*

8 + ··

When we add those functions, we get a real analytic function:

1 1 (2 − *e*−*ix*)+ (2 − *eix*) 4 − 2 cos *x*

2 − *eix* + 2 − *e*−*ix* = (2 − *eix*)(2 − *e*−*ix*) = 5 − 4 cos *x* (28)

This ratio is the inﬁnitely smooth function whose cosine coeﬃcients are 1*/*2*k*.

Example 6 Find *ck*

for the 2*π*-periodic shifted pulse *F* (*x*) = 1 for *s* ≤ *x* ≤ *s* + *h*

0 elsewhere in [−*π, π*]

.

Solution The integrals (26) from −*π* to *π* become integrals from *s* to *s* + *h*:

∫

Σ Σ

. Σ

1

*ck* = 2*π*

s + h

s

1 · *e*−ikx

1

*dx* =

2*π*

*e*−*ikx* s + h



−*ik* s

= *e*−*iks*

1 − *e*−*ikh*



2*πik*

*.* (29)

*Notice above all the simple eﬀect of the shift by s*. *It “modulates” each ck by e*−*iks*. The energy is unchanged, the integral of |*F* |2 just shifts, and all |*e*−*iks*| = 1:

Shift *F* (*x*) to *F* (*x* − *s*) ←→ Multiply *ck* by *e*−*iks.* (30)

Example 7 Centered pulse with shift *s* = −*h/*2. The square pulse becomes centered around *x* = 0. This even function equals 1 on the interval from −*h/*2 to *h/*2:

*h ikh/*2 1 − *e*−*ikh*

 

1 sin(*kh/*2)



Centered by *s* = − 2 *ck* = *e*

=

2*πik* 2*π*

*.*

*k/*2

Divide by *h* for a tall pulse. The ratio of sin(*kh/*2) to *kh/*2 is the sinc function:

∞

Tall pulse *F*centered = 1 Σ

*h*

sinc . *kh*Σ *eikx* = . 1*/h* for − *h/*2 ≤ *x* ≤ *h/*2

That division by *h* produces area = 1. Every coeﬃcient approaches 1 as *h* → 0.

2*π*

2*π*

−∞

2

0 elsewhere in [−*π, π*]

The Fourier series for the tall thin pulse again approaches the Fourier series for *δ*(*x*).

Hilbert space can containΣvectors *c*∫= (*c*0*, c*1*, c*−1*, c*2*, c*−2*,* ··· ) instead of functions

2

2

*F* (*x*). The length of *c* is 2*π*

|*ck*| =

|*F* | *dx*. The function space is often denoted





by *L*2 and the vector space is *A*2. The energy identity is trivial (but deep). Integrating

the Fourier series for *F* (*x*) times *F* (*x*), orthogonality kills every *cnck* for *n* ƒ= *k*. This

leaves the *ckck* = |*ck*|2:

∫

∫

*π*

|*F* (*x*)| *dx* =

2

−*π*

*π*

( *cneinx*)(

Σ

Σ

−*π*

*cke*−*ikx*)*dx* = 2*π*(|*c*0|2 + |*c*1|2 + |*c*−1|2 + ··) *.* (31)

*This is Plancherel’s identity: The energy in x-space equals the energy in k-space*.

Σ

Finally I want to emphasize the three big rules for operating on *F* (*x*) = *ckeikx*:

*dF*

1. The derivative has Fourier coeﬃcients *ikck* (energy moves to high *k*). *dx*
2. The integral of *F* (*x*) has Fourier coeﬃcients *ck ,k* = 0 (faster decay).

ƒ

*ik*

1. The shift to *F* (*x*− *s*) has Fourier coeﬃcients *e*−*iksck* (no change in energy).

# Application: Laplace’s Equation in a Circle

Our ﬁrst application is to Laplace’s equation. The idea is to construct *u*(*x, y*) as an inﬁnite series, choosing its coeﬃcients to match *u*0(*x, y*) along the boundary. Every- thing depends on the shape of the boundary, and we take a circle of radius 1.

Begin with the simple solutions 1, *r* cos *θ*, *r* sin *θ*, *r*2 cos 2*θ*, *r*2 sin 2*θ*, ... to Laplace’s equation. Combinations of these special solutions give all solutions in the circle:

(32)

*u*(*r, θ*) = *a*0 + *a*1*r* cos *θ* + *b*1*r* sin *θ* + *a*2*r*2 cos 2*θ* + *b*2*r*2 sin 2*θ* + ···

It remains to choose the constants *ak* and *bk* to make *u* = *u*0 on the boundary.

For a circle *u*0(*θ*) is periodic, since *θ* and *θ* + 2*π* give the same point:

Set *r* = 1 *u*0(*θ*) = *a*0 + *a*1 cos *θ* + *b*1 sin *θ* + *a*2 cos 2*θ* + *b*2 sin 2*θ* + ··· (33)

This is exactly the Fourier series for *u*0. The constants *ak* and *bk* must be the Fourier coeﬃcients of *u*0(*θ*). Thus the problem is completely solved, if an inﬁnite series (32) is acceptable as the solution.

Example 8 Point source *u*0 = *δ*(*θ*) at *θ* = 0 The whole boundary is held at *u*0 = 0, except for the source at *x* = 1, *y* = 0. Find the temperature *u*(*r, θ*) inside.

∞

1

Fourier series for *δ u* (*θ*) =

0

2*π*

1 1

+ (cos *θ* + cos 2*θ* + cos 3*θ* + ··· ) =

*π*

2*π*

Σ *einθ*

−∞

Inside the circle, each cos *nθ* is multiplied by *rn*:

1

Inﬁnite series for *u u*(*r, θ*) =

2*π*

+ 1 (*r* cos *θ* + *r*2 cos 2*θ* + *r*3 cos 3*θ* + ··· ) (34)

Poisson managed to sum this inﬁnite series! It involves a series of powers of *reiθ*.

*π*

So we know the response at every (*r, θ*) to the point source at *r* = 1, *θ* = 0:



Temperature inside circle

1

*u*(*r, θ*) =

2*π* 1+ *r*2 − 2*r* cos *θ*

2

1 − *r*

(35)

At the center *r* = 0, this produces the average of *u*0 = *δ*(*θ*) which is *a*0 = 1*/*2*π*. On the

boundary *r* = 1, this produces *u* = 0 except at the point source where cos 0 = 1:

1 1 − *r*2

1 1+ *r*



On the ray *θ* = 0 *u*(*r, θ*) = 2*π* 1+ *r*2 − 2*r* = 2*π* 1 − *r.* (36)

As *r* approaches 1, the solution becomes inﬁnite as the point source requires.

Example 9 Solve for any boundary values *u*0(*θ*) by integrating over point sources. When the point source swings around to angle *ϕ*, the solution (35) changes from *θ* to *θ* − *ϕ*. Integrate this “Green’s function” to solve in the circle:



1

Poisson’s formula *u*(*r, θ*) =

2*π*

∫

*π*

*u*0(*ϕ*)

2

1 − *r*

1+ *r*2 − 2*r* cos(*θ* − *ϕ*)

*dϕ*

(37)

−*π*

Ar *r* = 0 the fraction disappears and *u* is the average *u*0(*ϕ*)*dϕ/*2*π*. The steady state temperature at the center is the average temperature around the circle.

Poisson’s formula illustrates a key idea. Think of any *u*0(*θ*) as a circle of point sources. The source at angle *ϕ* = *θ* produces the solution inside the integral (37). Integrating around the circle adds up the responses to all sources and gives the response to *u*0(*θ*).

∫

Example 10 *u*0(*θ*) = 1 on the top half of the circle and *u*0 = −1 on the bottom half.

Solution The boundary values are the square wave *SW* (*θ*). Its sine series is in (8):

Square wave for *u* (*θ*) *SW* (*θ*) = 4 Σ sin *θ* + sin 3*θ* + sin 5*θ* + ··· Σ (38)

0

*π*

1

3

5

Inside the circle, multiplying by *r*, *r*2, *r*3,... gives fast decay of high frequencies:

4 Σ *r* sin *θ*

Rapid decay inside *u*(*r, θ*) =

*π*

+

1



*r*3 sin 3*θ*



*r*5 sin 5*θ* Σ



Laplace’s equation has smooth solutions, even when *u*0(*θ*) is not smooth.

+

3

5 + ···

(39)

# WORKED EXAMPLE

A hot metal bar is moved into a freezer (zero temperature). The sides of the bar are coated so that heat only escapes at the ends. *What is the temperature u*(*x, t*) along the bar at time *t*? It will approach *u* = 0 as all the heat leaves the bar.

Solution The heat equation is *ut* = *uxx*. At *t* = 0 the whole bar is at a constant temperature, say *u* = 1. The ends of the bar are at zero temperature for all time *t>* 0. This is an initial-boundary value problem:

Heat equation *ut* = *uxx* with *u*(*x,* 0) = 1 and *u*(0*, t*) = *u*(*π, t*) = 0*.* (40)

Those zero boundary conditions suggest a sine series. Its coeﬃcients depend on *t*:

∞

Σ

Series solution of the heat equation *u*(*x, t*) = *bn*(*t*) sin *nx.* (41)

1

The form of the solution shows separation of variables. In a comment below, we look for products *A*(*x*) *B*(*t*) that solve the heat equation and the boundary conditions. What we reach is exactly *A*(*x*) = sin *nx* and the series solution (41).

Two steps remain. First, choose each *bn*(*t*) sin *nx* to satisfy the heat equation:

Substitute into *ut* = *uxx bn*j (*t*) sin *nx* = −*n*2*bn*(*t*) sin *nx*

*bn*(*t*) = *e*−*n t bn*(0).

2

Notice *bn*j = *n*2*bn*. Now determine each *bn*(0) from the initial condition *u*(*x,* 0) = 1

on (0*, π*). Those numbers are the Fourier sine coeﬃcients of *SW* (*x*) in equation (38):



Box function/square wave

Σ

∞

*b* (0) sin *nx* = 1

4

*n*

*b* (0) = for odd *n*

*n*

1

*πn*

−

This completes the series solution of the initial-boundary value problem:

Bar temperature *u*(*x, t*) =

Σ

odd *n*

4 *e*−*n t* sin *nx.* (42)

*πn*

2

For large *n* (high frequencies) the decay of *e*−*n*2*t* is very fast. The dominant term (4*/π*)*e*−*t* sin *x* for large times will come from *n* = 1. This is typical of the heat equation and all diﬀusion, that the solution (the temperature proﬁle) becomes very smooth as *t* increases.

*Numerical diﬃculty* I regret any bad news in such a beautiful solution. To compute *u*(*x, t*), we would probably truncate the series in (42) to *N* terms. When that ﬁnite series is graphed on the website, serious bumps appear in *uN* (*x, t*). You ask if there is a physical reason but there isn’t. The solution should have maximum temperature at the midpoint *x* = *π/*2, and decay smoothly to zero at the ends of the bar.

Those unphysical bumps are precisely the Gibbs phenomenon. The initial *u*(*x,* 0) is 1 on (0*, π*) but its odd reﬂection is −1 on (−*π,* 0). That jump has produced the slow 4*/πn* decay of the coeﬃcients, with Gibbs oscillations near *x* = 0 and *x* = *π*. The sine series for *u*(*x, t*) is not a success numerically. Would ﬁnite diﬀerences help?

Separation of variables We found *bn*(*t*) as the coeﬃcient of an eigenfunction sin *nx*. Another good approach is to put *u* = *A*(*x*) *B*(*t*) directly into *ut* = *uxx*:

Separation *A*(*x*) *B* j(*t*) = *A* jj(*x*) *B*(*t*) requires *A* jj(*x*) = *B* j(*t*) = constant*.* (43)

*A*(*x*) *B*(*t*)

*A* jj*/A* is constant in space, *B* j*/B* is constant in time, and they are equal:

*A* jj = −*λ* gives *A* = sin √*λx* and cos √*λx B* j = −*λ* gives *B* = *e*−*λt*

*A*

*B*

The products *AB* = *e*−*λt* sin √*λx* and *e*−*λt* cos √*λx* solve the heat equation for any number *λ*. But the boundary condition *u*(0*, t*) =√0 eliminates the cosines. Then

*u*(*π, t*) = 0 requires *λ* = *n*2 = 1*,* 4*,* 9*,...* to have sin *λπ* = 0. Separation of variables

has recovered the functions in the series solution (42).

Finally *u*(*x,* 0) = 1 determines the numbers 4*/πn* for odd *n*. We ﬁnd zero for even *n* because sin *nx* has *n/*2 positive loops and *n/*2 negative loops. For odd *n*, the extra positive loop is a fraction 1*/n* of all loops, giving slow decay of the coeﬃcients.

Heat bath (the opposite problem) The solution on the website is 1 *u*(*x, t*), because it solves a diﬀerent problem. The bar is initially frozen at *U* (*x,* 0) =

−

0. It is placed into a heat bath at the ﬁxed temperature *U* = 1 (or *U* = *T*0). The new unknown is *U* and its boundary conditions are no longer zero.

The heat equation and its boundary conditions are solved ﬁrst by *UB*(*x, t*). In this example *UB* ≡ 1 is constant. Then the diﬀerence *V* = *U* − *UB* has zero boundary values, and its initial values are *V* = −1. Now the eigenfunction method (or sepa- ration of variables) solves for *V* . (The series in (42) is multiplied by −1 to account for *V* (*x,* 0) = −1.) Adding back *UB* solves the heat bath problem: *U* = *UB* + *V* = 1 − *u*(*x, t*).

Here *UB* ≡ 1 is the *steady state* solution at *t* = ∞, and *V* is the *transient* solution.

The transient starts at *V* = −1 and decays quickly to *V* = 0.

*Heat bath at one end* The website problem is diﬀerent in another way too. The

Dirichlet condition *u*(*π, t*) = 1 is replaced by the Neumann condition *u* j(1*, t*) = 0. Only the left end is in the heat bath. Heat ﬂows down the metal bar and out at the far end, now located at *x* = 1. How does the solution change for ﬁxed-free?

Again *UB* = 1 is a steady state. The boundary conditions apply to *V* = 1 − *UB*:



2

Fixed-free eigenfunctions

*V* (0) = 0 and *V* j(1) = 0 lead to *A*(*x*) = sin .*n* + 1 Σ *πx.* (44)

Those eigenfunctions give a new form for the sum of *Bn*(*t*) *An*(*x*):

Fixed-free solution *V* (*x, t*) = Σ *B* (0) *e*−(*n*+ 1 )2 *π*2 *t* sin .*n* + 1 Σ *πx.* (45)

odd *n*

*n*

2

2

All frequencies shift by 1 and multiply by *π*, because *A* jj = −*λA* has a free end

at *x* = 1. The crucial question is: Does orthogonality still hold for these new

2

. Σ

eigenfunctions sin *n* + 1 *πx* on [0*,* 1]? The answer is *yes* because this ﬁxed-free “Sturm–Liouville problem” *A* jj = −*λA* is still symmetric.

2

Summary The series solutions all succeed but the truncated series all fail. We can see the overall behavior of *u*(*x, t*) and *V* (*x, t*). But their exact values close to the jumps are not computed well until we improve on Gibbs.

We could have solved the ﬁxed-free problem on [0*,* 1] with the ﬁxed-ﬁxed solution on [0*,* 2]. That solution will be symmetric around *x* = 1 so its slope there is zero. Then rescaling *x* by 2*π* changes sin(*n* + 1 )*πx* into sin(2*n* + 1)*x*. I hope you like the

2

graphics created by Aslan Kasimov on the cse website.