**UNIT - 3**

**Differentiating Vector Functions of a Single Variable**

Your experience of differentiation and integration has extended as far as *scalar*

functions of single and multiple variables — functions like *f* (*x* ) and *f* (*x, y, t*).

It should be no great surprise that we often wish to differentiate vector func- tions. For example, suppose you were driving along a wiggly road with position **r**(*t*) at time *t*. Differentiating **r**(*t*) wrt time should yield your velocity **v**(*t*), and differentiating **v**(*t*) should yield your acceleration. Let’s see how to do this.



#### Differentiation of a vector

The derivative of a vector function **a**(*p*) of a single parameter *p* is

**a**′(*p*) = lim

*δp*→0

**a**(*p* + *δp*) − **a**(*p*) *.*

*δp*

If we write **a** in terms of components relative to a FIXED coordinate system (ˆ*ı,*ˆ*, k*ˆ*k*

constant)

**a**(*p*) = *a*1(*p*)ˆ*ı* + *a*2(*p*)ˆ** + *a*3(*p*)*k*ˆ

then

**a**′(*p*) = *da*1ˆ*ı* + *da*2ˆ** + *da*3 *k*ˆ*k .*

*dp dp dp*

That is, in order to differentiate a vector function, one simply differentiates each component separately. This means that all the familiar rules of differentiation apply, and they don’t get altered by vector operations like scalar product and vector products.

Thus, for example:

*d d* **a** *d* **b** *d d* **a** *d* **b**

*dp* (**a** × **b**) =

*dp* × **b** + **a** × *dp*

*dp* (**a** · **b**) =

*dp* · **b** + **a** · *dp .*

Note that *d* **a***/dp* has a different direction and a different magnitude from **a**.

Likewise, as you might expect, the chain rule still applies. If **a** = **a**(*u*) and *u* = *u*(*t*), say:

*d*

*dt* **a** =

*d* **a** *du du dt*

♣ Examples



**Q** A 3D vector **a** of constant magnitude is varying over time. What can you say about the direction of **a˙**?

**A** Using intuition: if only the direction is changing, then the vector must be tracing out points on the surface of a sphere. We would guess that the

derivative **a˙** is orthogonal to **a**.

To prove this write

*d d* **a** *d* **a** *d* **a**

*dt* (**a** · **a**) = **a** · *dt* + *dt* · **a** = 2**a** · *dt .*

But (**a** · **a**) = *a*2 which we are told is constant. So

*d d* **a**

*dt* (**a** · **a**) = 0 ⇒ 2**a** · *dt* = 0

and hence **a** and *d* **a***/dt* must be perpendicular.

**Q** The position of a vehicle is **r**(*u*) where *u* is the amount of fuel consumed by some time *t*. Write down an expression for the acceleration.

**A** The velocity is

*d* **r**

**v** = =

*dt*

*d* **r** *du du dt*

2 2 2

*dt*

*d d* **r** *d*

**a** = =

*dt dt*

*du*2

**r** .*du* Σ

*d* **r** *d u*

+

*du dt*2



* + 1. Geometrical interpretation of vector derivatives

Let **r**(*p*) be a position vector tracing a space curve as some parameter *p* varies. The vector *δ***r** is a secant to the curve, and *δ***r***/δp* lies in the same direction. (See Fig. 3.1.) In the limit as *δp* tends to zero *δ***r***/δp* = *d* **r***/dp* becomes a tangent to the space curve. If the magnitude of this vector is 1 (i.e. a unit tangent), then

*d* **r** = *dp* so the parameter *p* is arc-length (metric distance). More generally, however, *p* will not be arc-length and we will have:

| |

*d* **r** *d* **r** *ds*

=

*dp ds dp*

So, the direction of the derivative is that of a **tangent to the curve**, and its magnitude is |*ds/dp*|, **the rate of change of arc length w.r.t the parameter**. Of course if that parameter *p* is time, the magnitude |*d* **r***/dt*| is the speed.

♣ **Example**



**Q** Draw the curve

*s*

**r** = *a* cos(√



*s*

)ˆ*ı* + *a* sin(√



*hs*

)ˆ** + √ *k*



ˆ

*a*2 + *h*2

*a*2 + *h*2

*a*2 + *h*2

where *s* is arc length and *h*, *a* are constants. Show that the tangent *d* **r***/ds* to the curve has a constant elevation angle w.r.t the *xy* -plane, and determine its magnitude.

A

*d* **r**

*ds* = −√



*a*

*a*2 + *h*2

sin ()ˆ*ı* + √

*a*

*a*2 + *h*2



cos ()ˆ** + √

*k*ˆ

*a*2 + *h*2



*h*

The project√ion on the *xy* plane has magnitude *a/*√*a*2 + *h*2 and in the *z*



direction *h/ a*2 + *h*2, so the elevation angle is a constant, tan−1(*h/a*).

We are expecting *d* **r***/ds* = 1, and indeed

.*a*2 sin2() + *a*2 cos2() + *h*2*/*√*a*2 + *h*2 = 1*.*



* + 1. Arc length is a special parameter!

It might seem that we can be completely relaxed about saying that any old pa- rameter *p* is arc length, but this is not the case. Why not? The reason is that arc length is special is that, whatever the parameter *p*,

*p*

. .

∫

*s* =

*p*0

*d* **r**

. *dp* . *dp .*

Perhaps another way to grasp the significance of this is using Pythagoras’ theorem on a short piece of curve: in the limit as *dx* etc tend to zero,

*ds*2 = *dx* 2 + *dy* 2 + *dz* 2 *.*



dp dp

**r** (p +  p)

d**r** ds

**r** (p)

 **r**



ds

d**r** 1

**r** (s +  s)

**r** (s)

 **r**

Figure 3.1: Left: *δ***r** is a secant to the curve but, in the limit as *δp* → 0, becomes a tangent. Right: if the parameter is arc length *s*, then |*d* **r**| = *ds*.

So if a curve is parameterized in terms of *p*



*ds dx* 2

.

= +

*dp dp*

*dy* 2

+

*dp*

*dz* 2

*.*

*dp*

As an example, suppose in our earlier example we had parameterized our helix as

**r** = *a* cos *p*ˆ*ı* + *a* sin *p*ˆ** + *hpk*ˆ

It would be easy just to *say* that *p* was arclength, but it would not be correct because



*ds dx* 2

.

= +

*dp dp*

*dy* 2

+

*dp*

*dz* 2



*dp*





= *a*2 sin2 *p* + *a*2 cos2 *p* + *h*2 = *a*2 + *h*2

. √

If *p* really was arclength, *ds/dp* = 1. So *p/*√*a*2 + *h*2 **is** arclength, not *p*.



#### Integration of a vector function

The integration of a vector function of a single scalar variable can be regarded simply as the reverse of differentiation. In other words

*p*2 *d* **a**(*p*)

∫

*dp*

*p*1 *dp*

For example the integral of the acceleration vector of a point over an interval of time is equal to the change in the velocity vector during the same time interval. However, many other, more interesting and useful, types of integral are possible, especially when the vector is a function of more than one variable. This requires the introduction of the concepts of scalar and vector fields.



#### Curves in 3 dimensions

In the examples above, parameter *p* has been either arc length *s* or time *t*. It doesn’t have to be, but these are the main two of interest. Later we shall look at some important results when differentiating w.r.t. time, but now let use look more closely at 3D curves defined in terms of arc length, *s*.

Take a piece of wire, and bend it into some arbitrary non-planar curve. This is a *space curve.* We can specify a point on the wire by specifying **r**(*s*) as a function of distance or arc length *s* along the wire.

* + 1. The Fr´enet-Serret relationships

We are now going to introduce a local orthogonal coordinate frame for each point *s* along the curve, ie one with its origin at **r**(*s*). To specify a coordinate frame we need three mutually perpendicular directions, and these should be *intrinsic* to the curve, not fixed in an external reference frame. The ideas were first suggested by two French mathematicians, F-J. Fr´enet and J. A. Serret.

* + - 1. Tangent ˆt

There is an obvious choice for the first direction at the point **r**(*s*), namely the

**unit tangent ˆt**. We already know that

**ˆt** = *d* **r**(*s*)

*ds*

* + - 1. Principal Normal nˆ

Recall that earlier we proved that if **a** was a vector of constant magnitude that varies in direction over time then *d* **a***/dt* was perpendicular to it. Because

**ˆt** has constant magnitude but varies over *s*, *d***ˆt***/d s* must be perpendicular to

**ˆt**.

Hence the principal normal **ˆn** is

*d***ˆt**

≥

= *κ***nˆ** : where *κ* 0 *.*

*ds*

*κ* is the **curvature**, and *κ* = 0 for a straight line. The plane containing **ˆt** and

**ˆn** is called the **osculating plane**.

* + - 1. The Binormal ˆb

The local coordinate frame is completed by defining the binormal

**ˆb**(*s*) = **ˆt**(*s*) × **nˆ**(*s*) *.*

Since **bˆ** · **ˆt** = 0,

*d* **ˆb** · **ˆt** + **bˆ** · *d***ˆt** = *d* **ˆb** · **ˆt** + **bˆ** · *κ***nˆ** = 0

*ds ds ds*

from which

*d* **ˆb ˆt** = 0*. ds*

·

But this means that *d* **ˆb***/d s* is along the direction of **nˆ**, or

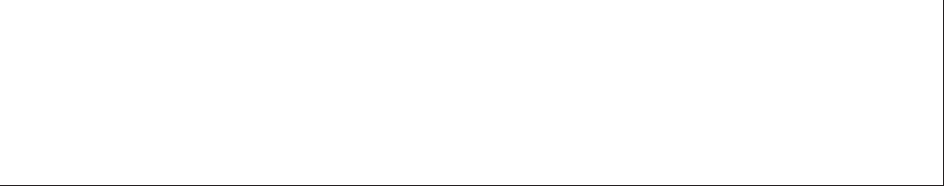
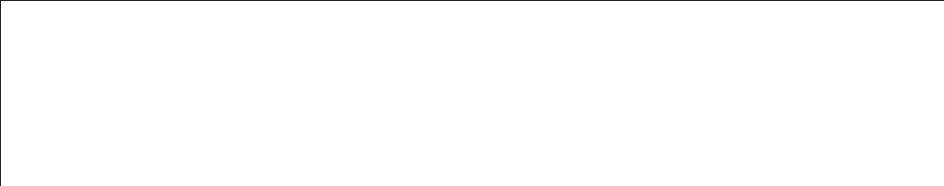
*d* **ˆb**

*ds* = −*τ*(*s*)**nˆ**(*s*)

where *τ* is the **torsion**, and the negative sign is a matter of convention. Differentiating **nˆ** · **ˆt** = 0 and **ˆn** · **bˆ** = 0, we find

*d* **ˆn** *ds*

= −*κ*(*s*)**ˆt**(*s*) + *τ*(*s*)**bˆ**(*s*)*.*



**The Fr´enet-Serret relationships:**

*d***ˆt***/d s* = *κ***nˆ**

*d* **ˆn***/d s* = −*κ*(*s*)**ˆt**(*s*) + *τ*(*s*)**bˆ**(*s*)

*d* **ˆb***/d s* = −*τ*(*s*)**nˆ**(*s*)

♣ **Example**



**Q** Derive *κ*(*s*) and *τ*(*s*) for the helix



*β β β*

and comment on their values.

**r**(*s*) = *a* cos . *s* Σˆ*ı* + *a* sin . *s* Σˆ** + *h* . *s* Σ *k*ˆ; *β* = √*a*2 + *h*2

**A** We found the unit tangent earlier as

**ˆt** = *d* **r**

*ds*

= Σ− *a* sin . *s* Σ *, a* cos . *s* Σ *, h* Σ *.*

Differentiation gives

*β*

*β*

*β*

*β*

*β*

ˆ

*κ***nˆ** = = Σ−

*d* **t** *a*

*β*2

*β*

*β*2

*β*



*ds*

cos . *s* Σ *,* − *a*

sin . *s* Σ *,* 0Σ

* 1. *RADIAL AND TANGENTIAL COMPONENTS IN PLANE POLARS*

Curvature is always positive, so

*κ* = *a* **nˆ** = Σ− cos . *s* Σ *,* − sin . *s* Σ *,* 0Σ *.*

*β*2

*β*

*β*

So the curvature is constant, and the normal is parallel to the *xy* -plane. Now use

ˆ ˆ .

ˆ*ı* ˆ**

ˆ*k*

. Σ *h*



. *s* Σ



*h* . *s* Σ *a* Σ

**b** = **t**×**ˆn** = (−*a/β*)*S* (*a/β*)*C* (*h/β*) =

. .

sin

*β*

*β ,* − *β* cos *β , β*

. −*C* −*S* 0 .

and differentiate **bˆ** to find an expression for the torsion

ˆ

*ds*

*β*2

*d* **b** = Σ *h*

*β*

*β*2

*β*

*β*2

cos . *s* Σ *, h*

sin . *s* Σ *,* 0Σ = −*h* **nˆ**

so the torsion is

*h*

*τ* = *β*2

again a constant.



#### Radial and tangential components in plane polars

In plane polar coordinates, the radius vector of any point *P* is given by

**r** = *r* cos *θ*ˆ*ı* + *r* sin *θ*ˆ**



ˆ**

P

*r*

*θ*

ˆ*ı*

= *r* **ˆe***r*

where we have introduced the unit radial vec- tor

**ˆe***θ*

**ˆe***r*

**ˆe***r* = cos *θ*ˆ*ı* + sin *θ*ˆ* .*

The other “natural” (we’ll see why in a later lecture) unit vector in plane polars is orthog- onal to **ˆe***r* and is

**ˆe***θ* = − sin *θ*ˆ*ı* + cos *θ*ˆ**

so that **ˆe***r* · **ˆe***r* = **ˆe***θ* · **ˆe***θ* = 1 and **ˆe***r* · **ˆe***θ* = 0.

Now suppose *P* is moving so that **r** is a function of time *t*. Its velocity is

*d*

**r˙** =



(*r* **ˆe**

*dr*

) = **ˆe**



+ *r d***ˆe***r*



*dt r dt r dt*

*dr dθ*

= **ˆe** + *r* (− sin *θ*ˆ*ı* + cos *θ*ˆ**)

*dt r dt*

*dr dθ*

= *dt* **ˆe***r* + *r dt* **ˆe***θ*

= radial + tangential

The radial and tangential components of velocity of *P* are therefore *dr/dt* and

*rdθ/dt*, respectively.

Differentiating a second time gives the acceleration of *P*

*d* 2*r*

*dr dθ*

*dr dθ*

*d* 2*θ*

*dθ dθ*

**¨r** =

*dt*2 **ˆe***r* + *dt dt* **ˆe***θ* + *dt dt* **ˆe***θ* + *r dt*2 **ˆe***θ* − *r dt dt* **ˆe***r*

Σ*d* 2*r*

2

=

*dt*2 − *r*

**ˆe***r* +

2 *dt dt* + *r dt*2

**ˆe***θ*

.*dθ* Σ2Σ



*dt*

Σ *dr dθ*



*d θ* Σ



#### Rotating systems

Consider a body which is rotating with constant angular velocity *ω* about some axis passing through the origin. Assume the origin is fixed, and that we are sitting in a fixed coordinate system *Oxyz* .

If *ρ* is a vector of constant magnitude and constant direction in the rotating system, then its representation **r** in the fixed system must be a function of *t*.

**r**(*t*) = R(*t*)*ρ*

At any instant as observed in the fixed system

*d* **r** = R˙ *ρ* + R*ρ*˙

*dt*

but the second term is zero since we assumed *ρ* to be constant so we have

*d* **r** = R˙ R⊤**r**

*dt*

Note that:

* *d* **r***/dt* will have fixed magnitude;
* *d* **r***/dt* will always be perpendicular to the axis of rotation;
* *d* **r***/dt* will vary in direction within those constraints;
* **r**(*t*) will move in a plane in the fixed system.

# 





Now let’s consider the term

R˙ R⊤. First, note that RR⊤ = I (the identity), so

differentiating both sides yields

R˙ R⊤ + RR˙ ⊤ = 0

R˙ R⊤ = −RR˙ ⊤

Thus R˙ R⊤ is anti-symmetric:

R˙ R⊤ = 

0 −*z y*

*z* 0 −*x*



−*y x* 0 

Now you can verify for yourself that application of a matrix of this form to an arbitrary vector has precisely the same effect as the cross product operator, *ω* , where *ω* = [*xyz* ]⊤. Loh-and-behold, we then we have

×

**r**˙ = *ω* × **r**

matching the equation at the end of lecture 2, **v** = *ω*×**r**, as we would hope/expect.

* + 1. Rotation: Part 2

Now suppose *ρ* is the position vector of a point *P* which **moves** in the rotating frame. There will be two contributions to motion with respect to the fixed frame, one due to its motion within the rotating frame, and one due to the rotation itself. So, returning to the equations we derived earlier:

**r**(*t*) = R(*t*)*ρ*(*t*)

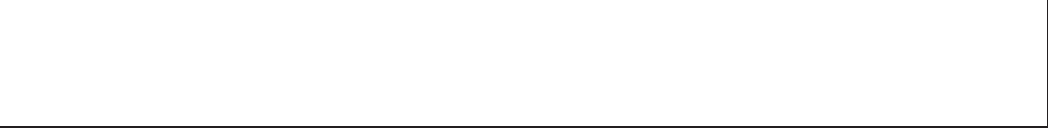
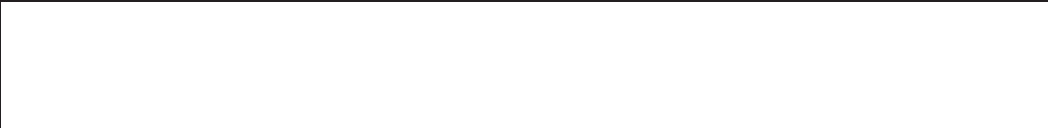
and the instantaenous differential with respect to time:

*d* **r** = R˙ *ρ* + R*ρ*˙

*dt*

= R˙ R⊤**r** + R*ρ*˙

Now *ρ* is not constant, so its differential is not zero; hence rewriting this last equations we have that



The **instantaneous velocity** of *P* in the fixed frame is

= R*ρ*˙ + *ω* × **r**

*dt*

*d* **r**

The second term of course, is the contribution from the rotating frame which we

saw previously. The first is the linear velocity measured in the rotating frame *ρ*˙,

referred to the fixed frame (via the rotation matrix R which aligns the two frames)

P at *t+*  *t*

  **r**



P at *t*

 **r)**  *t*

**r=**  *at t*

* + 1. Rotation 3: Instantaneous acceleration

Our previous result is a general one relating the time derivatives of any vector in rotating and non-rotating frames. Let us now consider the second differential:

¨**r** = *ω*˙

× **r** + *ω* × **r**˙ + R˙ *ρ*˙

+ R*ρ*¨*ρ*

We shall assume that the angular acceleration is zero, which kills off the first term,

and so now, substituting for **r**˙ we have

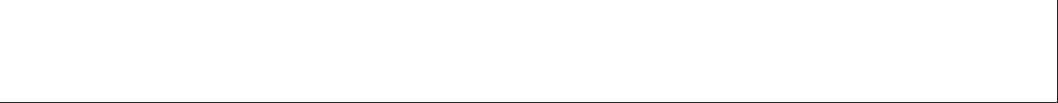
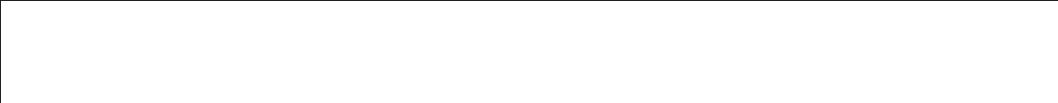
¨**r** = *ω* × (*ω* × **r** + R*ρ*˙) + R˙ *ρ*˙ + R*ρ*¨*ρ*

= *ω* × (*ω* × **r**) + *ω* × R*ρ*˙ + R˙ *ρ*˙ + R*ρ*¨*ρ*

= *ω* × (*ω* × **r**) + *ω* × R*ρ*˙ + R˙ (R⊤R)*ρ*˙

+ R*ρ*¨*ρ*

= *ω* × (*ω* × **r**) + 2*ω* × (R*ρ*˙) + R*ρ*¨*ρ*



The **instantaneous acceleration** is therefore

**¨r** = R*ρ*¨*ρ* + 2*ω* × (R*ρ*˙) + *ω* × (*ω* × **r**)

The first term is the acceleration of the point *P* in the rotating frame mea- sured in the rotating frame, but referred to the fixed frame by the rotation R

•

* + - * The last term is the centripetal acceleration to due to the rotation. (Yes! Its magnitude is *ω*2*r* and its direction is that of −**r**. Check it out.)

*γt*



*ω* = *ω***mˆ**

**r**

**mˆ**

**ˆ***ℓ*

**nˆ** *γ* = *γ***ˆ***ℓ*

Figure 3.2: Coriolis example.

The middle term is an extra term which arises because of the velocity of *P* in the rotating frame. It is known as the **Coriolis acceleration**, named after the French engineer who first identified it.

•

Because of the rotation of the earth, the Coriolis acceleration is of great im- portance in meteorology and accounts for the occurrence of high pressure anti- cyclones and low pressure cyclones in the northern hemisphere, in which the Coriolis acceleration is produced by a pressure gradient. It is also a very important compo- nent of the acceleration (hence the force exerted) by a rapidly moving robot arm, whose links whirl rapidly about rotary joints.

♣ **Example**



**Q** Find the instantaneous acceleration of a projectile fired along a line of longi- tude (with angular velocity of *γ* constant relative to the sphere) if the sphere is rotating with angular velocity *ω*.

**A** Consider a coordinate frame defined by mutually orthogonal unit vectors,

**ˆ***ℓ,* **mˆ** and **nˆ**, as shown in Fig. 3.2. We shall assume, without loss of generality,

that the fixed and rotating frames are instantaneously aligned at the moment shown in the diagram, so that R = I, the identity, and hence **r** = *ρ*.

In the rotating frame

*ρ*˙ = *γ* × *ρ* and *ρ*¨*ρ* = *γ* × *ρ*˙ = *γ* × (*γ* × *ρ*)

So the in the fixed reference frame, because these two frames are instanta- neously aligned

**¨r** = *γ* × (*γ* × *ρ*) + 2*ω* × (*γ* × *ρ*) + *ω* × (*ω* × **r**) *.*

The first term is the centripetal acceleration due to the projectile moving around the sphere — which it does because of the gravitational force. The

last term is the centripetal acceleration resulting from the rotation of the sphere. The middle term is the Coriolis acceleration.

Using Fig. 3.2, at some instant *t*

and

**r**(*t*) = *ρ*(*t*) = *r* cos(*γt*)**mˆ**

*γ* = *γ***ˆ***ℓ*

+ *r* sin(*γt*)**nˆ**

Then

*γ* × (*γ* × *ρ*) = (*γ* · *ρ*)*γ* − *γ*2*ρ* = −*γ*2*ρ* = −*γ*2**r***,*

Check the direction — the negative sign means it points *towards* the centre of the sphere, which is as expected.

Likewise the last term can be obtained as

*ω* × (*ω* × **r**) = −*ω*2*r* sin(*γt*)**nˆ**

Note that it is perpendicular to the axis of rotation minus sign, directed towards the axis)

The Coriolis term is derived as:

**mˆ** , and because of the

2*ω* × *ρ*˙ = 2*ω* × (*γ* × *ρ*)

0





0

= 2 ×

*ω*

*γ*

0

×

*r* cos *γt*

0

0

 *r* sin *γt* 

= 2*ωγr* cos *γt***ˆ***ℓ*

Instead of a projectile, now consider a rocket on rails which stretch north from the equator. As the rocket travels north it experiences the Coriolis force (exerted by the rails):

2 *γ ω R* cos *γt* **ˆ***ℓ*

+ve -ve +ve +ve

Hence the coriolis force is in the direction opposed to **ˆ***ℓ* (i.e. in the opposite direction to the earth’s rotation). In the absence of the rails (or atmosphere) the rocket’s tangetial speed (relative to the surface of the earth) is *greater* than the speed of the surface of the earth underneath it (since the radius of successive lines of latitude decreases) so it would (to an observer on the earth) appear to deflect to the east. The rails provide a coriolis force keeping it on the same meridian.



**Vector Operators: Grad, Div and Curl**

In the first lecture of the second part of this course we move more to consider properties of fields. We introduce three field operators which reveal interesting collective field properties, viz.

* + - * the **gradient** of a scalar field,
      * the **divergence** of a vector field, and
      * the **curl** of a vector field.

There are two points to get over about each:

The mechanics of taking the grad, div or curl, for which you will need to brush up your multivariate calculus.

•

The underlying physical meaning — that is, why they are worth bothering about.

•

In Lecture 6 we will look at combining these vector operators.

#### The gradient of a scalar field

Recall the discussion of temperature distribution throughout a room in the overview, where we wondered how a scalar would vary as we moved off in an arbitrary direc- tion. Here we find out how.

If *U*(*x, y, z* ) is a scalar field, ie a scalar function of position **r** = [*x, y, z* ] in 3 dimensions, then its **gradient** at any point is defined in Cartesian co-ordinates by

grad*U* = *∂U*ˆ*ı* + *∂U*ˆ** + *∂U k*ˆ*k .*

*∂x ∂y ∂z*

It is usual to define the **vector operator** which is called “del” or “nabla”

= ˆ*ı ∂*

∇

*∂x*

+ ˆ* ∂*

*∂y*

+ *k*ˆ*k ∂ .*

*∂z*

Then

grad*U* ≡ ∇*U .*

Note immediately that ∇*U* is a vector field!

Without thinking too carefully about it, we can see that the gradient of a scalar field tends to point in the direction of greatest change of the field. Later we will be more precise.

♣ Worked examples of gradient evaluation



1. *U* = *x* 2

⇒ ∇*U* = . *∂* ˆ*ı* + *∂* ˆ** + *∂ k*ˆ*k* Σ *x* 2 = 2*x*ˆ*ı .*

*∂x*

*∂y*

*∂z*



1. *U* = *r* 2

*r* 2 = *x* 2 + *y* 2 + *z* 2

⇒ ∇*U* = . *∂* ˆ*ı* + *∂* ˆ** + *∂ k*ˆ Σ (*x* 2 + *y* 2 + *z* 2)

*∂x*

*∂y*

*∂z*

= 2*x*ˆ*ı* + 2*y*ˆ** + 2*z k*ˆ = 2 **r** *.*



1. *U* = **c** · **r**, where **c** is constant.

⇒ ∇*U* = .ˆ*ı ∂*

*∂x*

*∂y*

*∂z*

3

1

2

3

+ ˆ* ∂*

+ *k*ˆ*k ∂*

Σ (*c*

*x* + *c*

1

*y* + *c*

2

*z* ) = *c* ˆ*ı* +*c* ˆ** +*c*

*k*ˆ*k* = **c** *.*





1. *U* = *f* (*r* ), where *r* = (*x* 2 + *y* 2 + *z* 2)

*U* is a function of *r* alone so *df /dr* exists. As *U* = *f* (*x, y, z* ) also,

√

*∂f df ∂r*

=

*∂f df ∂r*

=

*∂f df ∂r*

= *.*

*∂x dr ∂x*

*∂y dr ∂y*

*∂z dr ∂z*

⇒ ∇*U* = *∂f* ˆ*ı* + *∂f* ˆ** + *∂f k*ˆ*k* = *df* . *∂r* ˆ*ı* + *∂r* ˆ** + *∂r k*ˆ*k*Σ

*∂x*

*∂y*

*∂z*

*dr*

*∂x*

*∂y*

*∂z*



But *r* = √*x* 2 + *y* 2 + *z* 2, so *∂r/∂x* = *x/r* and similarly for *y, z* .

*df*

**r**

⇒ ∇*U* =

*dr*

*r*

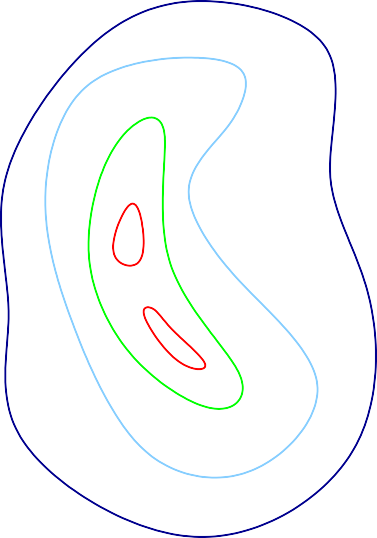
*dr*

*r*

.*x*ˆ*ı* + *y*ˆ** + *z k*ˆΣ = *df*

. Σ *.*





grad*U*

**r**

*U*(**r**)

*d* **r**

**r** + *d* **r**

*U*(**r** + *d* **r**)

Figure 5.1: The directional derivative

#### The significance of grad

If our current position is **r** in some scalar field *U* (Fig. 5.1), and we move an infinitesimal distance *d* **r**, we know that the change in *U* is

*d U* =

*∂U*

*dx* +

*∂x*

*∂U*

*dy* +

*∂y*

*∂U*

*dz .*

*∂z*

But we know that *d* **r** = (ˆ*ıd x* + ˆ*d y* + *k*ˆ*d z* ) and *U* = (ˆ*ı∂U/∂x* + ˆ*∂U/∂y* +

∇

*k*ˆ*∂U/∂z*), so that the change in *U* is also given by the scalar product

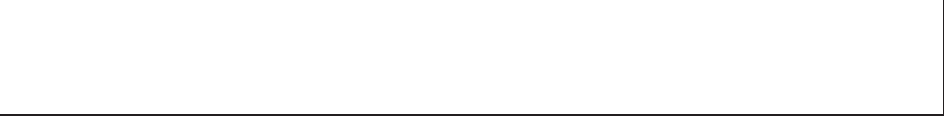
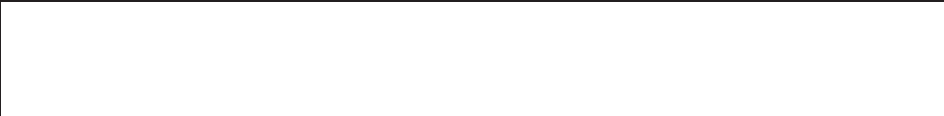
*d U* = ∇*U* · *d* **r** *.*

Now divide both sides by *ds*

*d U d* **r**

*ds* = ∇*U* · *ds .*

But remember that |*d* **r**| = *ds*, so *d* **r***/ds* is a unit vector in the direction of *d* **r**. This result can be paraphrased as:

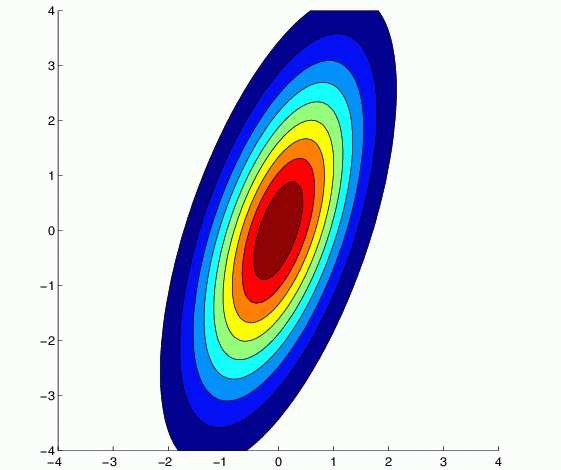
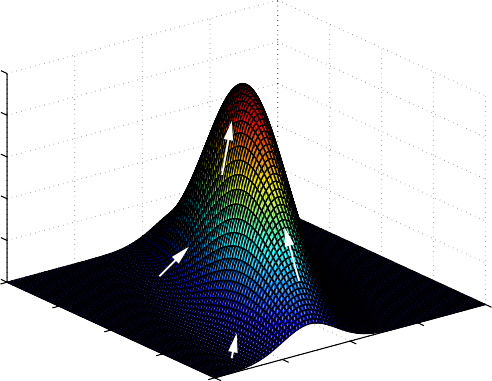
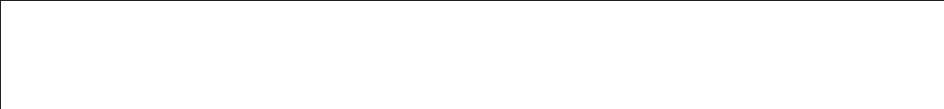


grad*U* has the property that the rate of change of *U* wrt distance in a particular direction (**dˆ**) is the projection of grad*U* onto that direction (or the component of grad*U* in that direction).

•

The quantity *d U/ds* is called a **directional derivative**, but note that in general it has a different value for each direction, and so has no meaning until you specify the direction.

We could also say that



* At any point P, grad*U* points in the direction of greatest change of

*U* at P, and has magnitude equal to the rate of change of *U* wrt

distance in that direction.

0.1

0.08

0.06

0.04

0.02

0

4

2

4

0

2

0

−2

−2

−4 −4

Another nice property emerges if we think of a surface of constant *U* – that is the locus (*x, y, z* ) for

*U*(*x, y, z* ) = constant *.*

If we move a tiny amount within that iso-*U* surface, there is no change in *U*, so

*d U/ds* = 0. So for any *d* **r***/ds* in the surface

*d* **r**

∇*U* · *ds* = 0 *.*

But *d* **r***/ds* is a tangent to the surface, so this result shows that

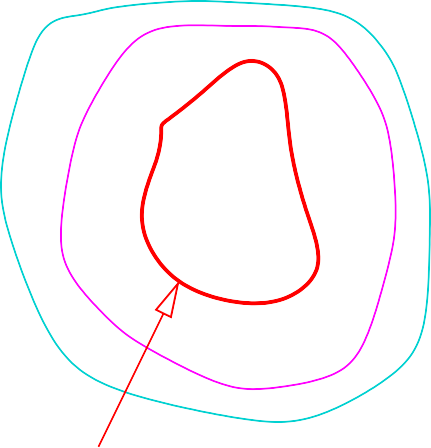
* + - grad*U* is everywhere NORMAL to a surface of constant *U*.



gradU

###### Surface of constant U

These are called Level Surfaces



Surface of constant U

#### The divergence of a vector field

The divergence computes a scalar quantity from a vector field by differentiation. If **a**(*x, y, z* ) is a vector function of position in 3 dimensions, that is **a** = *a*1ˆ*ı* + *a*2ˆ** +

*a*3*k*ˆ*k*, then its divergence at any point is defined in Cartesian co-ordinates by

div**a** = *∂a*1 + *∂a*2 + *∂a*3

*∂x ∂y ∂z*

We can write this in a simplified notation using a scalar product with the vector differential operator:

∇

div**a** = ˆ*ı ∂*

.

*∂x*

+ ˆ* ∂*

*∂y*

+ *k*ˆ*k ∂*

*∂z*

Σ · **a** = ∇ · **a**

Notice that the divergence of a vector field is a scalar field.

♣ Examples of divergence evaluation

**a** div**a**

1) *x*ˆ*ı* 1

2) **r**(= *x*ˆ*ı* + *y*ˆ** + *z k*ˆ) 3

3) **r***/r* 3 0

4) *r* **c**, for **c** constant (**r** · **c**)*/r*



We work through example 3).

The *x* component of **r***/r* 3 is *x.*(*x* 2 + *y* 2 + *z* 2)−3*/*2, and we need to find *∂/∂x* of it.

*∂ x.*(*x* 2 + *y* 2 + *z* 2)−3*/*2 = 1*.*(*x* 2 + *y* 2 + *z* 2)−3*/*2 + *x* −3(*x* 2 + *y* 2 + *z* 2)−5*/*2*.*2*x*

*∂x* 2

= *r* −3

.1 − 3*x* 2*r* −2Σ *.*

The terms in *y* and *z* are similar, so that

div(**r***/r* 3) = *r* −3

= 0

.3 − 3(*x* 2 + *y* 2 + *z* 2)*r* −2Σ

= *r* −3 (3 − 3)



#### The significance of div

Consider a typical vector field, water flow, and denote it by **a**(**r**). This vector has magnitude equal to the mass of water crossing a unit area perpendicular to the direction of **a** per unit time.

Now take an infinitesimal volume element *d V* and figure out the balance of the flow of **a** in and out of *d V* .

To be specific, consider the volume element *d V* = *dxdydz* in Cartesian co- ordinates, and think first about the face of area *dxdz* perpendicular to the *y* axis and facing outwards in the negative *y* direction. (That is, the one with surface area *d* **S** = −*dxdz*ˆ**.)

z



dS = -dxdz **j**

*dz*

dS = +dxdz **j**

y

*dx*

*dy*

*x*

Figure 5.2: Elemental volume for calculating divergence.

The component of the vector **a** normal to this face is **a** ˆ** = *ay* , and is pointing inwards, and so its contribution to the OUTWARD flux from this surface is

·

**a** · *d* **S** = − *ay* (*y* )*dzdx ,*

where *ay* (*y* ) means that *ay* is a function of *y* . (By the way, flux here denotes mass per unit time.)

A similar contribution, but of opposite sign, will arise from the opposite face, but we must remember that we have moved along *y* by an amount *dy* , so that this OUTWARD amount is

*ay* (*y* + *dy* )*dzdx* = .*ay*

+ *∂ay dy dxdz*

*∂y*

Σ

The total outward amount from these two faces is

*∂ay dydxdz* = *∂ay d V*

*∂y ∂y*

Summing the other faces gives a total outward flux of

*∂ax*

.



*∂x*

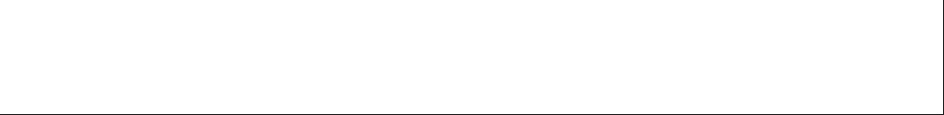
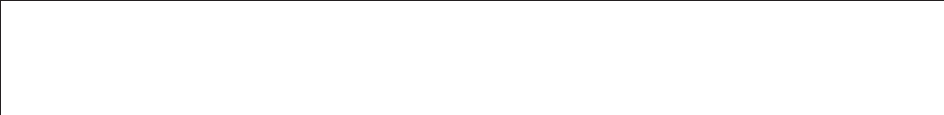
+ *∂ay*

*∂y*

+ *∂az* Σ *d V* = ∇ · **a** *d V*

So we see that

*∂z*



The divergence of a vector field represents the flux generation per unit volume at each point of the field. (Divergence because it is an efflux not an influx.)

Interestingly we also saw that the total efflux from the infinitesimal volume was equal to the flux integrated over the surface of the volume.

(NB: The above does not constitute a rigorous proof of the assertion because we have not proved that the quantity calculated is independent of the co-ordinate system used, but it will suffice for our purposes.)



* 1. **The Laplacian:** div(grad*U*) **of a scalar field**

Recall that grad*U* of *any* scalar field *U* is a vector field. Recall also that we can compute the divergence of any vector field. So we can certainly compute div(grad*U*), even if we don’t know what it means yet.

Here is where the ∇ operator starts to be really handy.

∇ · (∇*U*) = .ˆ*ı ∂*

*∂x*

*∂x*

*∂z*

+ ˆ* ∂*

*∂y*

+ *k*ˆ

*∂* Σ · ..ˆ*ı ∂*

+ ˆ* ∂*

*∂y*

*∂*

+ *k*ˆ *∂ U*

*∂z*

Σ Σ

= ˆ*ı ∂*

..

*∂x*

2

*∂x*

.

*∂z*

+ ˆ* ∂* +

*∂y*

2 2

Σ

*k*ˆ Σ · .ˆ*ı ∂*

+ ˆ* ∂*

*∂y*

+ *k*ˆ *∂ U*

*∂z*

ΣΣ

= *∂*

*∂x* 2

2

*∂*

+ *∂y* 2

2

*∂*

+ *∂z* 2 *U*

2

Σ

= .*∂*

*∂x* 2

*U ∂ U ∂ U*

+ +

*∂y* 2

*∂z* 2

This last expression occurs frequently in engineering science (you will meet it next in solving Laplace’s Equation in partial differential equations). For this reason, the operator ∇2 is called the “Laplacian”

∇2*U* = . *∂* + *∂* + *∂* Σ *U*

*∂x* 2

*∂y* 2

*∂z* 2

2

2

2

Laplace’s equation itself is

∇2*U* = 0

♣ Examples of ∇

2

*U* **evaluation**

*U* ∇2*U*



1) *r* 2(= *x* 2 + *y* 2 + *z* 2) 6

1. *xy* 2*z* 3 2*xz* 3 + 6*xy* 2*z*

3) 1*/r* 0



Let’s prove example (3) (which is particularly significant – can you guess why?).

1*/r* = (*x* 2 + *y* 2 + *z* 2)−1*/*2

*∂ ∂* (*x* 2 + *y* 2 + *z* 2)−1*/*2 = *∂*

− *x.*(*x* 2 + *y* 2 + *z* 2)−3*/*2

*∂x ∂x*

*∂x*

= −(*x* 2 + *y* 2 + *z* 2)−3*/*2 + 3*x.x.*(*x* 2 + *y* 2 + *z* 2)−5*/*2

= (1*/r* 3)(−1 + 3*x* 2*/r* 2)

Adding up similar terms for *y* and *z*

2

1 1

∇2 =

*r*

*r* 3

.−3 + 3(*x*

+ *y* 2

*r* 2

+ *x* 2

)Σ = 0

#### The curl of a vector field

So far we have seen the operator ∇ applied to a scalar field ∇*U*; and dotted with a vector field ∇ · **a**.

We are now overwhelmed by an irrestible temptation to

* + cross it with a vector field ∇ × **a**

This gives the **curl of a vector field**

∇ × **a** ≡ curl(**a**)

We can follow the pseudo-determinant recipe for vector products, so that

ˆ

ˆ*ı* ˆ* k*

. .



∇ × **a** =

*∂ ∂ ∂*

. *∂x ∂y ∂z* .

.

.

(remember it this way)

*ax ay az*

*∂y*

*∂z*

*∂z*

*∂x*

*∂x*

*∂y*



= .*∂az* − *∂ay* Σˆ*ı* + .*∂ax* − *∂az* Σˆ** + .*∂ay* − *∂ax* Σ *k*ˆ

♣ Examples of curl evaluation

**a** ∇ × **a**



1) −*y*ˆ*ı* + *x*ˆ** 2*k*ˆ

2) *x* 2*y* 2*k*ˆ*k* 2*x* 2*y*ˆ*ı* − 2*xy* 2ˆ**



#### The significance of curl

Perhaps the first example gives a clue. The field **a** = *y*ˆ*ı* + *x*ˆ** is sketched in Figure 5.3(a). (It is the field you would calculate as the velocity field of an object rotating with *ω* = [0*,* 0*,* 1].) This field has a curl of 2*k*ˆ, which is in the r-h screw sense out of the page. You can also see that a field like this must give a finite

−

value to the line integral around the complete loop H*C* **a** · *d* **r***.*

*y*



*y*

*x*



y+dy

ax (y+dy)

dy

y x x+dx



dx

ay (x)

ay (x+dx)

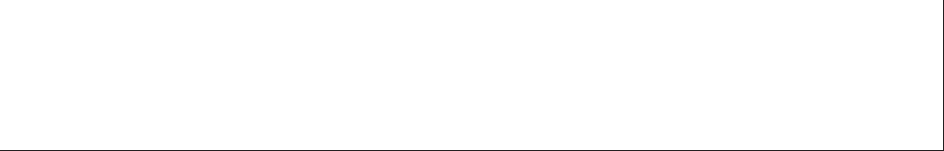
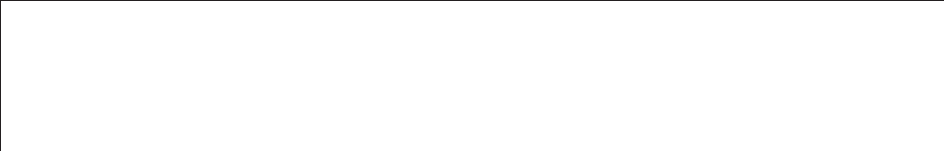


ax (y)

(a) (b)

Figure 5.3: (a) A rough sketch of the vector field −*y*ˆ*ı* + *x*ˆ**. (b) An element in which to calculate curl.

In fact curl is closely related to the line integral around a loop.



The **circulation** of a vector **a** round any closed curve *C* is defined to be

H

*C*

and the **curl** of the vector field **a** represents the **vorticity**, or **circulation**

**a** · *d* **r**

**per unit area**, of the field.

Our proof uses the small rectangular element *dx* by *dy* shown in Figure 5.3(b). Consider the circulation round the perimeter of a rectangular element.

The fields in the *x* direction at the bottom and top are

*ax* (*y* ) and *ax*

(*y* + *dy* ) = *ax*

(*y* ) + *∂ax dy,*

*∂y*

where *ax* (*y* ) denotes *ax* is a function of *y* , and the fields in the *y* direction at the left and right are

*ay* (*x* ) and *ay*

(*x* + *dx* ) = *ay*

(*x* ) + *∂ay dx*

*∂x*

Starting at the bottom and working round in the anticlockwise sense, the four contributions to the circulation *d C* are therefore as follows, where the minus signs take account of the path being opposed to the field:

*d C* = + [*ax* (*y* ) *dx* ] + [*ay* (*x* + *dx* ) *dy* ] − [*ax* (*y* + *dy* ) *dx* ] − [*ay* (*x* ) *dy* ]

= + [*ax* (*y* ) *dx* ] + Σ.*ay* (*x* ) + *dx* Σ *dy* Σ − Σ.*a* (*y* ) + *dy* Σ *dx* Σ − [*a* (*x* ) *dy* ]

*∂ay ∂ax*

*∂x*

*∂y*

*x*

*y*

= *∂ay*

.

Σ−

*∂x*

*∂ax dx dy*

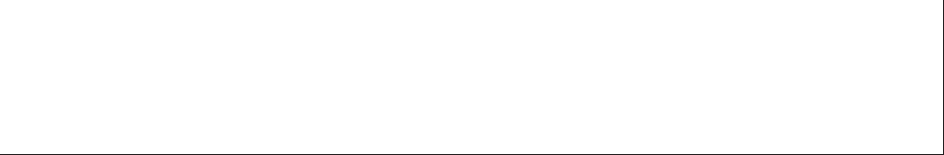
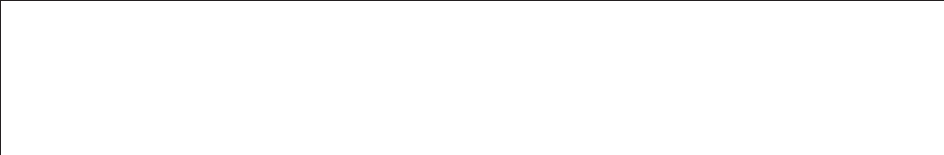
*∂y*

= (∇ × **a**) · *d* **S**

where *d* **S** = *dxdy k*ˆ*k*.

**NB:** Again, this is not a completely rigorous proof as we have not shown that the result is independent of the co-ordinate system used.

#### Some definitions involving div, curl and grad



* A vector field with zero divergence is said to be **solenoidal**.
* A vector field with zero curl is said to be **irrotational**.
* A scalar field with zero gradient is said to be, er, **constant**.

**Vector Operator Identities**

In this lecture we look at more complicated identities involving vector operators. The main thing to appreciate it that the operators behave both as vectors and as differential operators, so that the usual rules of taking the derivative of, say, a product must be observed.

There could be a cottage industry inventing vector identities. HLT contains a lot of them. So why not leave it at that?

First, since grad, div and curl describe key aspects of vectors fields, they arise often in practice, and so the identities can save you a lot of time and hacking of partial derivatives, as we will see when we consider Maxwell’s equation as an example later.

Secondly, they help to identify other practically important vector operators. So, although this material is a bit dry, the relevance of the identities should become clear later in other Engineering courses.

#### Identity 1: curl grad *U* = 0

ˆ*ı* ˆ* k*ˆ

.

.

∇ × ∇*U* = . *∂/∂x ∂/∂y ∂/∂z*

.

.

.

2

*∂U/∂x ∂U/∂y ∂U/∂z*

= ˆ*ı* . *∂ U*

*∂*2*U*

−

Σ + ˆ** () + *k*ˆ ()

*∂y∂z*

= 0 *,*

*∂z∂y*

as *∂*2*/∂y∂z* = *∂*2*/∂z∂y* .

Note that the output is a null *vector*.

#### Identity 2: div curl a = 0

∇ · ∇ × **a** =

*∂/∂x ∂/∂y ∂/∂z*

*∂/∂x ∂/∂y ∂/∂z ax ay az*

.

.

.

.

.

.

*∂*2*az*

*∂*2*ay*

*∂*2*az*

*∂*2*ax*

*∂*2*ay*

*∂*2*ax*

= *∂x∂y* − *∂x∂z* − *∂y∂x* + *∂y∂z* + *∂z∂x* − *∂z∂y*

= 0



#### Identity 3: div and curl of *U*a

Suppose that *U*(**r**) is a scalar field and that **a**(**r**) is a vector field and we are inter- ested in the product *U***a**. This is a vector field, so we can compute its divergence and curl. For example the density *ρ*(**r**) of a fluid is a scalar field, and the instan- taneous velocity of the fluid **v**(**r**) is a vector field, and we are probably interested in mass flow rates for which we will be interested in *ρ*(**r**)**v**(**r**).

The divergence (a scalar) of the product *U***a** is given by:

∇ · (*U***a**) = *U*(∇ · **a**) + (∇*U*) · **a**

= *U*div**a** + (grad*U*) · **a**

In a similar way, we can take the curl of the vector field *U***a**, and the result should be a vector field:

∇ × (*U***a**) = *U* ∇ × **a** + (∇*U*) × **a** *.*



#### Identity 4: div of a × b

Life quickly gets trickier when vector or scalar products are involved: For example, it is not *that* obvious that

div(**a** × **b**) = curl**a** · **b** − **a** · curl**b** To show this, use the determinant:

*∂/∂xi ∂/∂xj ∂/∂xk ∂ ∂ ∂*

.

.



*ax ay az*

*bx by bz*

.

= *∂x* [*ay bz* − *az by* ] + *∂y* [*az bx* − *ax bz* ] + *∂z* [*ax by* − *ay bx* ]

= *. . .* bash out the products *. . .*

.

= curl**a** · **b** − **a** · (curl **b**)



* 1. **Identity 5:** curl(**a** × **b**)

ˆ*ı* ˆ* k*ˆ

. .

curl(**a** × **b**) = *∂/∂x ∂/∂y ∂/∂z*

.

.

*ay bz* − *az by az bx* − *ax bz ax by* − *ay bx*

.

.

so the ˆ*ı* component is

*∂ ∂*

*∂y* (*ax by* − *ay bx* ) − *∂z* (*az bx* − *ax bz* )

which can be written as the sum of four terms:

*a ∂by*

.

*x ∂y*

+ *∂bz b*

*∂z x*

Σ−

*∂ay*

*∂y*

.

+ *∂az* + *b*

*∂z*

Σ .

*∂*

*y ∂y*

*∂*

+ *bz ∂z*

Σ *ax*

−.*a*

*∂*

*y ∂y*

*∂*

+ *az ∂z*

Σ *bx*

Adding *ax* (*∂bx /∂x* ) to the first of these, and subtracting it from the last, and doing the same with *bx* (*∂ax /∂x* ) to the other two terms, we find that (you should of course check this):

∇ × (**a** × **b**) = (∇ · **b**)**a** − (∇ · **a**)**b** + [**b** · ∇]**a** − [**a** · ∇]**b**

where [**a** · ∇] can be regarded as new, and very useful, scalar differential operator.



#### Definition of the operator [a · ∇]

This is a *scalar operator*, but it can obviously can be applied to a scalar field, resulting in a scalar field, or to a vector field resulting in a vector field:

[**a** · ∇] ≡ Σ*a*

*∂*

*x ∂x*

*∂*

+ *ay ∂y*

*∂*

+ *az ∂z .*

Σ



* 1. **Identity 6:** curl(curl**a**) **for you to derive**

The following important identity is stated, and left as an exercise:

curl(curl**a**) = graddiv**a** − ∇2**a** where

∇2**a** = ∇2*ax*ˆ*ı* + ∇2*ay*ˆ** + ∇2*az k*ˆ



♣ Example of Identity 6: electromagnetic waves

**Q:** James Clerk Maxwell established a set of four vector equations which are fun- damental to working out how electromagnetic waves propagate. The entire telecommunications industry is built on these.

div**D** = *ρ*

div**B** = 0

*∂*

curl**E** = −*∂t* **B**

*∂*

curl**H** = **J** + **D**

*∂t*

In addition, we can assume the following, which should all be familiar to you:

**B** = *µr µ*0**H**, **J** = *σ***E**, **D** = *ǫr ǫ*0**E**, where all the scalars are constants.

Now show that in a material with zero free charge density, *ρ* = 0, and with zero conductivity, *σ* = 0, the electric field **E** must be a solution of the wave equation

∇ **E** = *µr µ*0*ǫr ǫ*0(*∂* **E***/∂t* ) *.*

2 2 2

**A:** First, a bit of respect. Imagine you are the first to do this — this is a tingle moment.

div**D** = div(*ǫr ǫ*0**E**) = *ǫr ǫ*0div**E** = *ρ* = 0 ⇒ div**E** = 0*.* (*a*) div**B** = div(*µr µ*0**H**) = *µr µ*0div**H** = 0 ⇒ div**B** = 0 (*b*) curl**E** = −*∂***B***/∂t* = −*µr µ*0(*∂***H***/∂t*) (*c* )

curl**H** = **J** + *∂***D***/∂t* = 0 + *ǫr ǫ*0(*∂***E***/∂t*) (*d* )

But we know (or rather you worked out in Identity 6) that curlcurl = graddiv

−

2

∇ , and using (c)

curlcurl**E** = graddiv**E** − ∇2**E** = curl (−*µr µ*0(*∂***H***/∂t*))

so interchanging the order of partial differentation, and using (a) div**E** = 0:

*∂*

2

−∇ **E** = −*µr µ*0 *∂t* (curl**H**)

= −*µ µ* .*ǫ ǫ*

*∂*



*∂***E** Σ

*r* 0 *∂t r*

2 *∂*2**E**



1. *∂t*

⇒ ∇ **E** = *µr µ*0*ǫr ǫ*0 *∂t*2

This equation is actually three equations, one for each component:

∇ *Ex* = *µr µ*0*ǫr ǫ*0

2

*∂*2*Ex*

*∂t*2

and so on for *Ey* and *Ez* .



#### Grad, div, curl and ∇2 in curvilinear co-ordinate systems

It is possible to obtain general expressions for grad, div and curl in any orthogonal curvilinear co-ordinate system by making use of the *h* factors which were introduced in Lecture 4.

We recall that the unit vector in the direction of increasing *u*, with *v* and *w* being kept constant, is

**u**ˆ =

1. *∂***r**



*hu ∂u*

where **r** is the position vector, and

*∂***r**

. .

*hu* = . *∂u* .

is the metric coefficient. Similar expressions apply for the other co-ordinate direc- tions. Then

*d* **r** = *hu***u**ˆ*d u* + *hv* ˆ**v***d v* + *hw* **w**ˆ*d w .*

#### Grad in curvilinear coordinates

Noting that *U* = *U*(**r**) and *U* = *U*(*u, v, w* ), and using the properties of the gradient of a scalar field obtained previously

*∂U ∂U ∂U*

∇*U* · *d* **r** = *d U* =

It follows that

*du* +

*∂u*

*dv* + *dw*

*∂v ∂w*

*∂U ∂U ∂U*

∇*U* · (*hu***u**ˆ*d u* + *hv* ˆ**v***d v* + *hw* **w**ˆ*d w* ) =

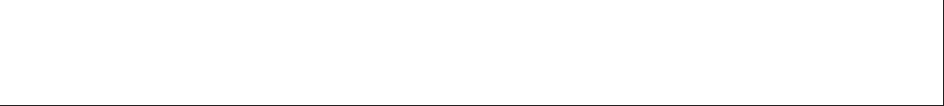
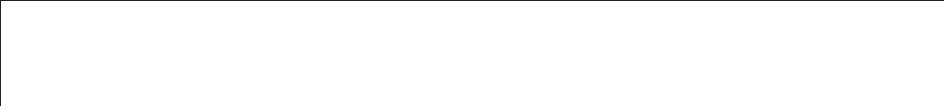
*du* +

*∂u*

*dv* + *dw*

*∂v ∂w*

The only way this can be satisfied for independent *du*, *dv* , *dw* is when



∇*U* = *h ∂u* **ˆu** + *h ∂v* **ˆv** + *h ∂w* **wˆ**

1 *∂U*

1 *∂U*

1 *∂U*

*u*

*v*

*w*

#### Divergence in curvilinear coordinates

Expressions can be obtained for the divergence of a vector field in orthogonal curvilinear co-ordinates by making use of the flux property.

We consider an element of volume *d V* . If the curvilinear coordinates are orthogonal then the little volume is a cuboid (to first order in small quantities) and

*d V* = *hu hv hw du dv dw .*

However, it is not quite a cuboid: the area of two opposite faces will differ as the scale parameters are functions of *u*, *v* and *w* in general.

*w*



##### h (v) dw

w

##### h (v+dv) dw

w

##### hu(v) du

*u*

*h*v *dv*

*y*

##### h (v+dv) du

u

The scale params are functions of u,v,w

Figure 6.1: Elemental volume for calculating divergence in orthogonal curvilinear coordinates

So the net efflux from the two faces in the **ˆv** direction shown in Figure 6.1 is

= Σ*av*

+ *∂av dv h*

*∂v u*

Σ Σ

+ *∂hu dv h*

*∂v w*

Σ Σ

+ *∂hw dv dudw a*

*∂v v*

Σ −

*huhw*

*dudw*

= *∂*(*av huhw* )*dudvdw*

*∂v*

which is easily shown by multiplying the first line out and dropping second order terms (i.e. (*dv* )2).

By definition div is the net efflux per unit volume, so summing up the other faces:

⇒ div**a** *huhv hw*

div**a** *d V* = *∂*(*au hv hw* )

*∂u*

.

*dudvdw* = *∂*(*au hv hw* )

.

*∂u*

+ *∂*(*av hu hw* )

*∂v*

+ *∂*(*av hu hw* )

*∂v*

+ *∂*(*aw hu hv* ) *dudvdw*

*∂w*

Σ

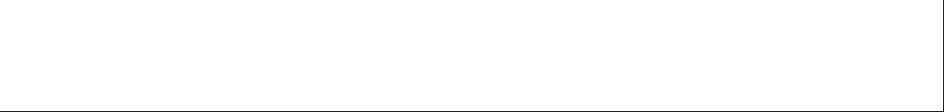
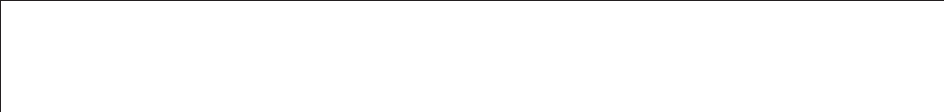
+ *∂*(*aw hu hv* ) *dudvdw*

Σ

*∂w*

*6.11. CURL IN CURVILINEAR COORDINATES* 79

So, finally,



1

div**a** = *h h h*

*u v w*

.

*∂*(*au hv hw* ) + *∂*(*av hu hw* ) + *∂*(*aw hu hv* )

*∂u ∂v ∂w*

Σ

#### Curl in curvilinear coordinates

Recall from Lecture 5 that we computed the *z* component of curl as the circulation per unit area from

*d C* = *∂ay*

.

*∂x*

*∂ax dx dy*

*∂y*

Σ−

By analogy with our derivation of divergence, you will realize that for an orthogonal curvilinear coordinate system we can write the area as *huhv dudw* . But the opposite sides are no longer quite of the same length. The lower of the pair in Figure 6.2 is length *hu*(*v* )*du*, but the upper is of length *hu*(*v* + *dv* )*du*

y v+dv



|  |  |
| --- | --- |
| au *(v+dv)* | |
| *h* u*(v+dv) du*  *dv*  *h*u*(v) du* |  |



*y* u *u+du*



au *(v)*



Figure 6.2: Elemental loop for calculating curl in orthogonal curvilinear coordinates

Summing this pair gives a contribution to the circulation

*au*(*v* )*hu*

(*v* )*du* − *au*

(*v* + *dv* )*hu*

(*v* + *dv* )*du* = *∂*(*huau* )*dvdu*

*∂v*

−

and together with the other pair:

*d C* = .−*∂*(*huau*) + *∂*(*hv av* )Σ *dudv*

*∂v*

*∂u*

So the circulation per unit area is

*d C* 1

=

*huhv dudv*

*huhv*

.*∂*(*hv av* ) − *∂*(*huau*)Σ

and hence curl is

*∂u*

*∂v*

1

curl**a**(*u, v, w* ) =

*hv hw*

.*∂*(*hw aw* ) − *∂*(*hv av* )Σ **uˆ** +

1 .*∂*(*huau*) − *∂*(*hw aw* )Σ **ˆv** +

*∂v*

*∂w*

*hw hu*

*∂w*

*∂u*

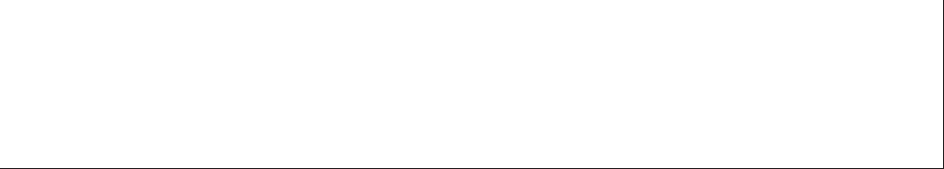
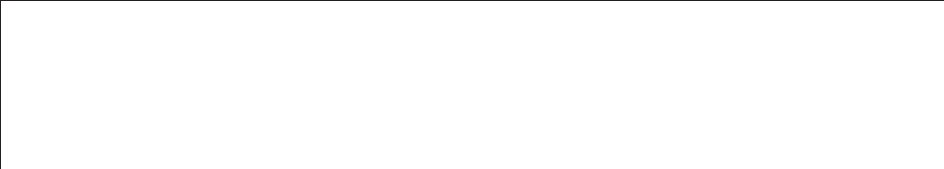
1 .*∂*(*hv av* ) − *∂*(*huau*)Σ **wˆ**

*huhv*

*∂u*

*∂v*

You should check that this can be written as



**Curl in curvilinear coords:**

curl**a**(*u, v, w* ) =

1

*h h h*

*u v w*

.

.

*hu***uˆ**

*∂*

*∂u*

*hv* **ˆv** *hw* **wˆ**

*∂ ∂*

*∂v ∂w*

.

*h*

*uau*

*h*

*v av*

*h*

*w aw*

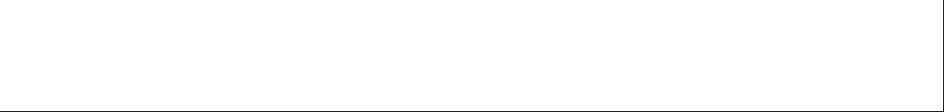
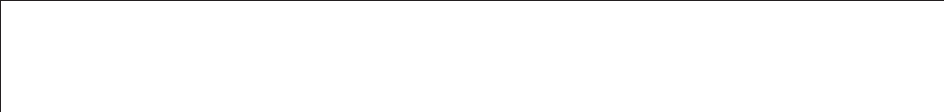
.

.

.

#### The Laplacian in curvilinear coordinates

Substitution of the components of grad*U* into the expression for div**a** immediately (!\*?) gives the following expression for the Laplacian in general orthogonal co- ordinates:



∇ *U* =

2

1

*huhv hw ∂u hu*

Σ .

*∂ hv hw ∂U*

*∂u*

Σ + .

*∂ hw hu ∂U*

*∂v hv*

*∂v*

Σ +

*∂ huhv ∂U*

*∂w hw ∂w*

.

ΣΣ *.*

#### Grad Div, Curl, ∇2 in cylindrical polars

Here (*u, v, w* ) → (*r, φ, z* ). The position vector is **r** = *r* cos *φ*ˆ*ı* + *r* sin *φ*ˆ** + *z k*ˆ, and

*hr* = |*∂***r***/∂r* |, etc.

⇒ *hr* = (cos2 *φ* + *sin*2*φ*) = 1*,*

.

√

*hφ* = (*r* 2 sin2 *φ* + *r* 2 cos2 *φ*) = *r, hz* = 1

*∂U*

⇒ grad*U* = **ˆe**

*∂r*

*r*

1 *∂U*

+ **ˆe**

*r*

*∂φ*

*φ*

+ *∂U k*ˆ*k*

div**a** = 1 .*∂*(*rar* ) + *∂aφ* Σ + *∂az*

*∂z*

*r*

*∂r*

*∂φ*

*∂z*

curl**a** = .1 *∂az*

*r*

*∂φ*

*∂z*

*∂r*

*φ*

*r*

*∂r*

*∂φ*

* *∂aφ* Σ **ˆe**

+ .*∂ar*

* *∂az* Σ **ˆe**

+ 1 .*∂*(*raφ*) − *∂ar* Σ *k*ˆ

∇ *U* = Tutorial Exercise

2

*∂z*

*r*

#### Grad Div, Curl, ∇2 in spherical polars

Here (*u, v, w* ) (*r, θ, φ*). The position vector is **r** = *r* sin *θ* cos *φ*ˆ*ı* + *r* sin *θ* sin *φ*ˆ** +

→

*r* cos *θk*ˆ.

⇒ *hr* = .(sin2 *θ*(cos2 *φ* + sin2 *φ*) + cos2 *θ*) = 1



*hθ* = .(*r* 2 cos2 *θ*(cos2 *φ* + sin2 *φ*) + *r* 2 sin2 *θ*) = *r hφ* = .(*r* 2 sin2 *θ*(sin2 *φ* + cos2 *φ*) = *r* sin *θ*

*∂U* 1 *∂U* 1 *∂U*

⇒ grad*U* =

*∂r* **ˆe***r* + *r*

*∂θ* **ˆe***θ* + *r* sin *θ ∂φ* **ˆe***φ*

1 *∂*(*r* 2*ar* )

1 *∂*(*aθ* sin *θ*)

1 *∂aφ*

div**a** =

*r* 2

+

*∂r r* sin *θ*

+

*∂θ r* sin *θ ∂φ*

curl**a** = **ˆe***r*

*r* sin *θ*

*∂*

*∂θ* (*aφ*

.

*∂*

sin *θ*) − *∂φ*

(*aθ*

) + **ˆe***θ*

*r* sin *θ*

Σ

*∂*

*∂φ*(*ar*

.

*∂*

) − *∂r*

(*aφ*

*r* sin *θ*)Σ +

**ˆe***φ* . *∂* (*a*

*r*

*∂r*

*∂*

*r* ) − (*a* )

*θ*

*∂θ*

*r*

Σ

∇ *U* = Tutorial Exercise

2



♣ Examples

**Q1** Find curl**a** in (i) Cartesians and (ii) Spherical polars when **a** = *x* (*x*ˆ*ı* + *y*ˆ** + *z k*ˆ).

**A1** (i) In Cartesians

ˆ*ı* ˆ* k*ˆ

.

. ˆ

curl**a** = *∂/∂x ∂/∂y ∂/∂z* = −*z*ˆ** + *y k .*

.

.

*x* 2 *xy xz*

.

.

(ii) In spherical polars, *x* = *r* sin *θ* cos *φ* and **r** = (*x*ˆ*ı* + *y*ˆ** + *z k*ˆ). So

**a** = *r* 2 sin *θ* cos *φ***ˆe***r*

⇒ *ar* = *r* sin *θ* cos *φ*; *aθ* = 0; *aφ* = 0 *.*

2

Hence as

*θ*

*r* sin *θ*

*φ*

*r*

*∂r*

*θ*

*∂θ*

*r*

curl**a** = **ˆe***r*

*r* sin *θ*

. *∂* (*a*

*∂*

sin *θ*) −

*∂θ*

*φ*

*∂φ*

(*a* )Σ+ **ˆe***θ*

*∂ ∂*

(*a* ) − (*a*

*∂φ*

*r*

*∂r*

.

*r* sin *θ*)Σ+ **ˆe***φ* . *∂* (*a r* ) − *∂* (*a* )Σ

curl**a** = **ˆe***θ* . *∂* (*r* 2 sin *θ* cos *φ*)Σ + **ˆe***φ* .− *∂*

(*r* 2 sin *θ* cos *φ*)Σ

*r* sin *θ ∂φ*

**ˆe***θ* 2



*r ∂θ*

**ˆe***φ* 2



= *r* sin *θ* (−*r*

sin *θ* sin *φ*) +

*r* .−*r* cos *θ* cos *φ*)Σ

= **ˆe***θ*(−*r* sin *φ*) + **ˆe***φ*(−*r* cos *θ* cos *φ*)

Checking: these two results should be the same, but to check we need ex- pressions for **ˆe***θ,* **ˆe***φ* in terms of ˆ*ı* etc.

Remember that we can work out the unit vectors **ˆe***r* and so on in terms of ˆ*ı*

etc using

1

ˆe =

*∂***r**

; **ˆe**

1 *∂***r**

= ; **ˆe**



1 *∂***r**

=

where **r** = *x*ˆ*ı* +*y*ˆ** +*z k*ˆ *.*

*r h*1 *dr*

*θ h*2 *dθ*

*φ h*3 *dφ*

Grinding through we find

**ˆe***r*



**ˆe***θ*

 =

sin *θ* cos *φ* sin *θ* sin *φ* cos *θ* ˆ*ı*

cos *θ* cos *φ* cos *θ* sin *φ* − sin *θ* ˆ**





ˆ*ı*

= [*R*] ˆ**





 **ˆe***φ* 

 − sin *φ* cos *φ* 0

  *k*ˆ 

 *k*ˆ 

Don’t be shocked to see a rotation matrix [*R*]: we are after all rotating one

right-handed orthogonal coord system into another.

So the result in spherical polars is

curl**a** = (cos *θ* cos *φ*ˆ*ı* + cos *θ* sin *φ*ˆ** − sin *θk*ˆ)(−*r* sin *φ*) + (− sin *φ*ˆ*ı* + cos *φ*ˆ**)(−*r* cos *θ* cos

= −*r* cos *θ*ˆ** + *r* sin *θ* sin *φk*ˆ

= −*z*ˆ** + *y k*ˆ*k*

which is exactly the result in Cartesians.



**Q2** Find the divergence of the vector field **a** = *r* **c** where **c** is a constant vector

1. using Cartesian coordinates and (ii) using Spherical Polar coordinates.

**A2** (i) Using Cartesian coords:

*∂*

div**a** =

*∂x*

(*x* 2 + *y* 2 + *z* 2)1*/*2*cx*

+ *. . .*

= *x.*(*x* 2 + *y* 2 + *z* 2)−1*/*2*cx* + *. . .*

1

= *r* **r** · **c** *.*

1. Using Spherical polars

**a** = *ar* **ˆe***r* + *aθ***ˆe***θ* + *aφ***ˆe***φ*

and our first task is to find *ar* and so on. We can’t do this by inspection, and finding their values requires more work than you might think! Recall

**ˆe***r*



**ˆe***θ*

 =

sin *θ* cos *φ* sin *θ* sin *φ* cos *θ* ˆ*ı*

cos *θ* cos *φ* cos *θ* sin *φ* − sin *θ* ˆ**





ˆ*ı*

= [*R*] ˆ**





 **ˆe***φ* 

 − sin *φ* cos *φ* 0

  *k*ˆ*k* 

 *k*ˆ 

Now the point is the same point in space whatever the coordinate system, so

*ar* **ˆe***r* + *aθ***ˆe***θ* + *aφ***ˆe***φ* = *ax*ˆ*ı* + *ay*ˆ** + *az k*ˆ and using the inner product

⊤





ˆ*ı*

 *ar* 

⊤

*θ*

 **ˆe***r* 

*θ*

 *ax* 

*y*

*a*

 *aφ* 

**ˆe** =

 **ˆe***φ* 

*a*

 *az*

ˆ**

 *k*ˆ 

 *ar*   *ax* 

⊤

⊤





ˆ*ı*



ˆ*ı*

*θ*

*y*

*a*

 *aφ* 

[*R*]

ˆ** =

*k*ˆ 

*a*

 *az*

ˆ**

 *k*ˆ 

*ar ax*

⊤

⊤













⇒ *aθ* [*R*] = *ay*

 *aφ* 

*ar a*



⇒

*θ*

⊤



 *aφ* 

 *az*

*ax*

⊤





= *a*

*y*

 *az*

[*R*]⊤

*ar ax*









⇒ *aθ* = [*R*] *ay*

 *aφ*   *az* 

For our particular problem, *ax* = *rcx* , etc, where *cx* is a constant, so now we can write down

*ar* = *r* (sin *θ* cos *φcx* + sin *θ* sin *φcy* + cos *θcz* ) *aθ* = *r* (cos *θ* cos *φcx* + cos *θ* sin *φcy* − sin *θcz* ) *aφ* = *r* (− sin *φcx* + cos *φcy* )

Now all we need to do is to bash out

1 *∂*(*r* 2*ar* )

1 *∂*(*aθ* sin *θ*)

1 *∂aφ*

div**a** =

*r* 2

+

*∂r r* sin *θ*

+

*∂θ r* sin *θ ∂φ*

In glorious detail this is

div**a** = 3 (sin *θ* cos *φcx* + sin *θ* sin *φcy* + cos *θcz* ) +

1 2 2

.



sin *θ* cos

1



*θ* − sin

*θ*)(cos *φcx* + sin *φcy* ) − 2 sin *θ* cos *θcz* Σ +

sin *θ* (− cos *φcx* − sin *φcy* )

A bit more bashing and you’ll find

div**a** = sin *θ* cos *φcx* + sin *θ* sin *φcy* + cos *θcz*

= **ˆe***r* · **c**

This is EXACTLY what you worked out before of course.



**Take home messages from these examples:**

Just as physical vectors are independent of their coordinate systems, so are differential operators.

•

Don’t forget about the vector geometry you did in the 1st year. Rotation matrices are useful!

•

Spherical polars were NOT a good coordinate system in which to think about this problem. Let the symmetry guide you.

•

**Gauss’ and Stokes’ Theorems**

This section finally begins to deliver on why we introduced div grad and curl. Two theorems, both of them over two hundred years old, are explained:

**Gauss’ Theorem** enables an integral taken over a volume to be replaced by one taken over the surface bounding that volume, and vice versa. Why would we want to do that? Computational efficiency and/or numerical accuracy!

•

**Stokes’ Law** enables an integral taken around a closed curve to be replaced by one taken over *any* surface bounded by that curve.

•



#### Gauss’ Theorem

Suppose that **a**(**r**) is a vector field and we want to compute the total flux of the field across the surface *S* that bounds a volume *V* . That is, we are interested in calculating:

∫ **a** · *d* **S**

*S*



d **S**

d **S**

d **S**

d **S**

d **S**

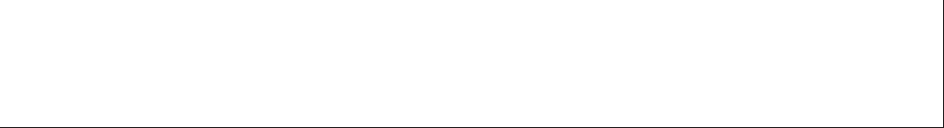
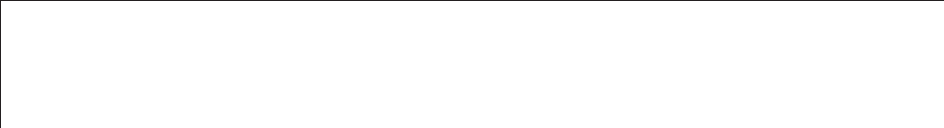
Figure 7.1: The surface element *d* **S** must stick out of the surface.

85

where recall that *d* **S** is normal to the locally planar surface element and **must everywhere point out of the volume** as shown in Figure 7.1.

Gauss’ Theorem tells us that we can do this by considering the total flux generated inside the volume *V* :

obtained by integrating the divergence over the entire volume.



**Gauss’ Theorem**

∫

*S*

**a** · *d* **S** = ∫ div **a** *d V*

*V*

#### Informal proof

An non-rigorous proof can be realized by recalling that we defined div by considering the efflux *d E* from the surfaces of an infinitesimal volume element

*d E* = **a** · *d* **S**

and defining it as

div **a** *d V* = *d E* = **a** · *d* **S** *.*

If we sum over the volume elements, this results in a sum over the surface elements. But if two elemental surface touch, their *d* **S** vectors are in opposing direction and cancel as shown in Figure 7.2. Thus the sum over surface elements gives the overall bounding surface.

Figure 7.2: When two elements touch, the *d* **S** vectors at the common surface cancel out. One can imagine building the entire volume up from the infinitesimal units.

♣ Example of Gauss’ Theorem

This is a typical example, in which the surface integral is rather tedious, whereas the volume integral is straightforward.

**Q** Derive *S* **a** *d* **S** where **a** = *z*

∫ ·

centred on the origin:

* + 1. directly;

3 ˆ*k* and *S* is the surface of a sphere of radius *R*

* + 1. by applying Gauss’ Theorem

3

z **k**



### 

R sin

R

d  d  **r**

dz

z

R

Figure 7.3:

**A** (1) On the surface of the sphere, **a** = *R*3 cos3 *θk*ˆ*k* and *d* **S** = *R*2 sin *θdθd φ***ˆr**.

Everywhere **ˆr** · *k*ˆ = cos *θ*.

⇒ ∫ **a** · *d* **S** = ∫

∫

2*π π*

*R*3 cos3 *θ . R*2 sin *θdθdφ***ˆe***r* · *k*ˆ*k*

*S φ*=0

∫

2*π*

=

*φ*=0

*θ*=0

*π*

∫

*θ*=0

*π*

∫

*R*3 cos3 *θ . R*2 sin *θdθdφ .* cos *θ*

= 2*πR*5

0

cos4 *θ* sin *θdθ*

2*πR*5



5 *π* 4*πR*5



= 5 Σ− cos *θ*Σ0 = 5

(2) To apply Gauss’ Theorem, we need to figure out div **a** and decide how to compute the volume integral. The first is easy:

div**a** = 3*z* 2

For the second, because div**a** involves just *z* , we can divide the sphere into discs of constant *z* and thickness *dz* , as shown in Fig. 7.3. Then

*d V* = *π*(*R*2 − *z* 2)*dz*

and

*R*

div **a***d V* = 3*π*

∫ ∫

−*R*

*V*

*z* 2(*R*2 − *z* 2)*dz*

= 3*π* Σ *R z*

*R*

2

3

5

* *z* Σ

3

4*πR*5

=

5

5 −*R*

